Collision operator for a strongly magnetized pure electron plasma

T.M. O'Neil
Department of Physics, University of California at San Diego, La Jolla, California 92093

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Collisions are discussed for a pure electron plasma which is so strongly magnetized that the Larmor radius is small compared to the classical distance of closest approach. In this regime, the most important collisions are characterized by an impact parameter which is the order of the distance of closest approach. Assuming that this distance is small compared to the interparticle spacing, a Boltzmann-like collision operator may be derived. In turn, this operator may be reduced to a Fokker–Planck operator, which is integrable. An interesting property of the collision operator results from an adiabatic invariant which is preserved during scattering. There is no energy exchange between the velocity components which are parallel to the field and the velocity components which are perpendicular to the field. This may have implications for a current effort to cool a pure electron plasma to low temperature by cyclotron radiation.

I. INTRODUCTION

In this paper we discuss the collision operator for a homogeneous, pure electron plasma which is strongly magnetized. In particular, the Larmor radius is assumed to be small compared to the classical distance of closest approach, that is $r_L \ll b$, where $r_L = \bar{v}/\Omega$ and $b = e^2/m\bar{v}^2$. Here, $\Omega = eB/mc$ is the cyclotron frequency and $\bar{v}$ is a characteristic electron velocity (e.g., thermal velocity for a Maxwellian plasma). To insure that the plasma is weakly correlated, the number of electrons in a Debye sphere is assumed to be large (i.e., $n\lambda_D^3 \gg 1$), and, to insure that the plasma is classical, the de Broglie wavelength is assumed to be small compared to the Larmor radius (i.e., $\hbar/m\bar{v} \ll r_L$, or equivalently, $\hbar\Omega \gg m\bar{v}^2$). Thus, the plasma is characterized by the ordering $\hbar/m\bar{v} \ll r_L \ll b \ll n^{-1/3} \ll \lambda_D$.

There has been much previous theoretical work on collisions in a magnetized plasma.[1–7] However, all of the previous work involves the assumption that a collision produces only a small perturbation in the orbit of an electron, and the derivation of the collision operator involves an integration along unperturbed electron orbits. We will see that this assumption, which may be called the assumption of weak interaction strength, is not appropriate in the regime $r_L \ll b$. The strong magnetic field effectively modifies the range of the interaction so that the most important collisions (as defined in Sec. III) are characterized by an impact parameter which is of order b, and the interaction strength for such collisions is not weak since $e^2/b = m\bar{v}^2$.

In this paper, we do not assume that the interaction strength is weak. Rather, advantage is taken of the short-range nature of the interaction, and a Boltzmann-like collision operator is derived from the BBGKY hierarchy.[8] Such a collision operator provides a valid description of well-separated binary collisions, and collisions characterized by an impact parameter of order b are of this nature. Recall that b is small compared to the interparticle spacing (i.e., $b \ll n^{-1/3}$).

In addition to modifying the effective range of the interaction between two electrons, the magnetic field restricts the possible outcomes of the interaction. An adiabatic invariant prevents the exchange of energy between the velocity components which are parallel to the magnetic field and the velocity components which are perpendicular to the field. This means that the collision operator does not force equipartition of energy between the parallel and perpendicular velocity components.

Since there is no exchange of energy between parallel and perpendicular velocity components, conservation of energy and conservation of momentum imply that the parallel velocity components for the two electrons are not changed by the interaction, or simply interchange (i.e., $v_{1z} \rightleftarrows v_{2z}$). This means that the distribution of parallel velocity components does not change in time.

The exchange of energy between the perpendicular velocity components for the two electrons is not restricted in this manner, and the distribution of perpendicular velocity components evolves to a Maxwellian. Moreover, the time evolution may be followed analytically. To be specific, the Boltzmann-like collision operator reduces to a Fokker–Planck operator, and the Fokker–Planck equation is integrable as an initial value problem.

In Sec. II, the scattering solution for the interaction of two electrons in a uniform magnetic field is obtained as an expansion in $1/B$. In Sec. III, the Boltzmann-like collision operator is derived from the BBGKY hierarchy. In Sec. IV, this operator is reduced to a Fokker–Planck operator, and the temporal evolution of the velocity distribution is discussed. In Appendix A we provide an alternate derivation of the adiabatic invariant and generalize the invariant to the case where many electrons interact simultaneously. In Appendix B, the collision operator derived in this paper is compared with that derived earlier by Rostoker.[1]

Before beginning the calculation, we note that the regime $r_L \ll b$ may be relevant to a current series of experiments.[9] That the regime is quite unusual can be seen by rewriting the inequality as $(kT)^{3/2} \ll 10^{-7} B$, where kT is in eV and B is in G. Even for $B = 100$ kG, the inequality is not satisfied for any temperature that would not lead to recombination. However, the experiments[9] alluded to involve the

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magnetic confinement of a pure electron plasma (nonneutral plasma), and recombination cannot occur for such a plasma. Moreover, there is an effort underway to cool a pure electron plasma down to the liquid and crystal states,[10] and the low temperature required (cryogenic range) is such that the plasma enters the regime $r_L \ll b$. The other two inequalities (i.e., $n\lambda_D^3 \gg 1$ and $\hbar\Omega \ll kT$) are satisfied, provided the temperature is not too low. In other words, there is an intermediate temperature regime where all three inequalities are satisfied.

Also, the results presented here may have implications for the cooling effort. A cooling mechanism such as cyclotron radiation reduces the perpendicular temperature but not the parallel temperature. Since the adiabatic invariant prevents energy exchange between the parallel and perpendicular degrees of freedom, one cannot rely on collisions to maintain equipartition of energy.

II. BINARY INTERACTION

Let us consider the electrostatic interaction of two electrons which move in the uniform magnetic field $\mathbf{B} = B\hat{z}$. In order to identify an adiabatic invariant which is preserved during the interaction, we introduce the sum and difference coordinates and velocities

$$\mathbf{R} = \mathbf{r}_2 + \mathbf{r}_1, \quad \mathbf{V} = \mathbf{v}_2 + \mathbf{v}_1, \qquad (1)$$
$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \quad \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1,$$

where $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are the coordinates and velocities of the two electrons. The sum and difference velocities satisfy the equations of motion

$$\frac{d\mathbf{V}}{dt} + \Omega\,\mathbf{V}\times\hat{z} = 0, \qquad (2)$$

$$\frac{d\mathbf{v}}{dt} + \Omega\mathbf{v}\times\hat{z} = \frac{2e^2}{m}\frac{\mathbf{r}}{|\mathbf{r}|^3}, \qquad (3)$$

where $\Omega = eB/mc$ is the electron-cyclotron frequency.

Let us choose $t = t_a$ to be some time before the electrons begin to interact and $t = t_b$ to be some time after the interaction is completed. From Eq. (2), it follows that

$$V_z(t_b) = V_z(t_a),$$
$$V_+(t_b) = V_+(t_a)e^{i\Omega(t_b - t_a)}, \qquad (4)$$

where $V_+ \equiv V_x + iV_y$.

Equation (3) is identical to the equation of motion for an electron which moves in a uniform magnetic field and in the electric field of a fixed charge. In the regions accessible to the electron, the scale length of the electric field is large compared with the Larmor radius of the electron. This follows from the inequality $r_L \ll b$. Thus, the adiabatic invariant $|v_+|^2/B$ is preserved during the interaction, and we can set $|v_+(t_b)| = |v_+(t_a)|$. Squaring both sides of the equation and adding it to the equation $|V_+(t_b)|^2 = |V_+(t_a)|^2$, which follows from Eq. (4), yields the result

$$|v_{2+}(t_b)|^2 + |v_{1+}(t_b)|^2 = |v_{2+}(t_a)|^2 + |v_{1+}(t_a)|^2, \qquad (5)$$

where we have set $V_+ = v_{2+} + v_{1+}$ and $v_+ = v_{2+} - v_{1+}$. In other words, the total energy associated with the perpendicular velocity components is conserved.

The underlying physics is the following. Because of its cyclotron motion, electron 1 produces a field of frequency Ω at the location of electron 2 and breaks the adiabatic invariant of electron 2 (i.e., $|v_{2+}|^2/B$), and vice versa. In this way, there is an exchange of energy between the perpendicular velocity components of electrons 1 and 2, but the total energy in the perpendicular velocity components is conserved. In Appendix A, this result is generalized to the case where many electrons interact simultaneously.

To calculate the exchange of energy between the perpendicular velocity components, we start by writing $v_+(t_b)$ in the form

$$v_+(t_b) = v_+(t_a)e^{i\Omega(t_b - t_a)}\,e^{-i\delta\theta}, \qquad (6)$$

where $\delta\theta$ is a phase shift. This form follows from $|v_+(t_b)| = |v_+(t_a)|$. The x and y components of Eq. (3) can be combined in the complex equation

$$\frac{dv_+}{dt} - i\Omega v_+ = \frac{2e^2}{m}\frac{x+iy}{|\mathbf{r}|^3}, \qquad (7)$$

which, when integrated, yields

$$v_+(t_b) = v_+(t_a)e^{i\Omega(t_b - t_a)} + \frac{2e^2}{m}\int_{t_a}^{t_b} dt'\,e^{i\Omega(t_b - t')}$$
$$\times [x(t') + iy(t')]/|\mathbf{r}(t')|^3 . \qquad (8)$$

We will find that $\delta\theta$ is small. Setting $e^{-i\delta\theta} \simeq 1 - i\delta\theta$ in Eq. (6), and comparing to Eq. (8), allows the identification

$$\delta\theta = \frac{2e^2 i}{mv_+(t_a)}\int_{t_a}^{t_b} dt'\,e^{i\Omega(t_a - t')}\frac{[x(t') + iy(t')]}{|\mathbf{r}(t')|^3}. \qquad (9)$$

To evaluate the integral in this equation, we express the electron orbit as the sum of a guiding-center motion plus cyclotron motion about the guiding center [i.e., $\mathbf{r}(t') = \mathbf{r}_g(t') + \delta\mathbf{r}(t')$], and we use the Taylor expansion

$$\frac{x(t') + iy(t')}{|\mathbf{r}(t')|^3} \simeq \frac{x_g(t') + iy_g(t')}{|\mathbf{r}_g(t')|^3} + \delta\mathbf{r}(t')\cdot\nabla_g$$
$$\times\frac{x_g(t') + iy_g(t')}{|\mathbf{r}_g(t')|^3}. \qquad (10)$$

The first term in the Taylor expansion gives negligible contribution to the integral, since the guiding-center motion is characterized by frequency components which are much smaller than Ω (i.e., $\bar{v}/b \ll \Omega$). Of course, the integral under discussion is simply the Fourier transform of expression (10) evaluated at the frequency Ω. The second term in the Taylor expansion gives the main contribution, since $\delta\mathbf{r}(t')$ varies at the frequency Ω. By using the expressions

$$\delta x(t') = \mathrm{Re}\{v_+(t_a)\exp[i\Omega(t' - t_a)]/i\Omega\},$$
$$\delta y(t') = \mathrm{Im}\{v_+(t_a)\exp[i\Omega(t' - t_a)]/i\Omega\}, \qquad (11)$$

we obtain (to first order in $1/B$)

$$\delta\theta = \frac{2e^2}{m\Omega}\int_{-\infty}^{+\infty} dt'\left(\frac{1}{[z_g^2(t') + \rho^2]^{3/2}} - \frac{3}{2}\frac{\rho^2}{[z_g^2(t') + \rho^2]^{5/2}}\right), \qquad (12)$$

where the t' integral has been extended to $\pm\infty$ and $\rho^2 = x_g^2 + y_g^2 = \mathrm{const}$.

This can be rewritten as

$$\delta\theta=\frac{4e^2}{m\Omega^2}\int_{z_{\min}}^{\infty}\frac{dz}{v_z(z)}\left(\frac{1}{(z^2+\rho^2)^{3/2}}-\frac{3}{2}\frac{\rho^2}{(z^2+\rho^2)^{5/2}}\right), \tag{13}$$

where $v_z(z)$ is given by

$$v_z(z)=\left(v_z^2(t_a)-\frac{4e^2/m}{(\rho^2+z^2)^{1/2}}\right)^{1/2}, \tag{14}$$

and $z_{\min}$ is the smaller of zero or the value of z where $v_z(z)=0$. In terms of scaled variables, $\delta\theta$ takes the form

$$\delta\theta=\frac{\alpha|v_z(t_a)|}{\rho\Omega}\int_{\sigma(\alpha)}^{\infty}d\xi \times\frac{[(\xi^2+1)^{-3/2}-\frac{3}{2}(\xi^2+1)^{-5/2}]}{[1-\alpha(\xi^2+1)^{-1/2}]^{1/2}}, \tag{15}$$

where $\alpha=4e^2/[m\rho v_z^2(t_a)]$ and $\sigma(\alpha)$ is zero for $\alpha\leqslant 1$ and $(\alpha^2-1)^{1/2}$ for $\alpha>1$. One may check that $|\delta\theta|\ll 1$, except for $\alpha\simeq 1$, where the integral diverges logarithmically and $|\delta\theta|\simeq-(m|v_z|^3/e^2\Omega)\ln|1-\alpha|\simeq r_L/b\,\ln|1-\alpha|$. In other words, $|\delta\theta|\ll 1$ except in an exponentially small range $|1-\alpha|\simeq\exp(-b/r_L)$. This range will be negligible in our future use of Eq. (15).

From Eqs. (1), (4), and (6), it follows that

$$|v_{2+}(t_b)|^2=|v_{2+}(t_a)+\tfrac{1}{2}(e^{-i\delta\theta}-1)[v_{2+}(t_a)-v_{1+}(t_a)]|^2, \tag{16}$$

which to second order in $\delta\theta$ is given by

$$\begin{aligned}|v_{2+}(t_b)|^2=&|v_{2+}(t_a)|^2+\delta\theta|v_{2+}(t_a)|\\&\times|v_{1+}(t_a)|\sin(\psi_2-\psi_1)+[(\delta\theta^2)^2/4]\\&\times[|v_{1+}(t_a)|^2-|v_{2+}(t_a)|^2].\end{aligned} \tag{17}$$

Here, the notation $v_{j+}=|v_{j+}|\exp(i\psi_j)$ has been introduced. The corresponding expression for $|v_{1+}(t_b)|^2$ is obtained by interchanging the subscripts 1 and 2 in Eq. (17). These are the desired expressions for the interchange of energy between the perpendicular velocity components for the two electrons. Of course, when the expression for $|v_{2+}(t_b)|^2$ is added to that for $|v_{1+}(t_b)|^2$, one simply recovers Eq. (5).

Turning next to a consideration of the parallel velocity components, we note that $|v_+(t_b)|^2=|v_+(t_a)|^2$ together with $|\mathbf{v}(t_b)|^2=|\mathbf{v}(t_a)|^2$ imply that $|v_z(t_b)|=|v_z(t_a)|$. This plus the result $V_z(t_b)=V_z(t_a)$ imply that the parallel components for the two electrons are unchanged by the interaction, or simply interchange. Recalling that $\alpha>1$ corresponds to reflection of one electron from the other and that $\alpha<1$ corresponds to no reflection, we set

$$\begin{aligned}&v_{1z}(t_b)=v_{1z}(t_a),\quad v_{2z}(t_b)=v_{2z}(t_a)\quad\text{for }\alpha<1,\\&v_{1z}(t_b)=v_{2z}(t_a),\quad v_{2z}(t_b)=v_{1z}(t_a)\quad\text{for }\alpha>1.\end{aligned} \tag{18}$$

III. BOLTZMANN-LIKE COLLISION OPERATOR

Generally speaking, a Boltzmann collision operator may be used to determine the effect on the particle distribution of well-separated binary collisions. It is not obvious that such an operator is appropriate for the case of electrons, which interact via the Coulomb interaction. This interaction is long range and typically leads to many electron effects such as Debye screening.

The reason a Boltzmann operator may be used is that the strong magnetic field effectively reduces the range of the interaction, at least for the scattering of the perpendicular velocity components. From Eq. (10) and the discussion following it, one can see that the perpendicular velocity components are scattered only by an interaction electric field with frequency components as high as the cyclotron frequency. For the ordering $\Omega\gg\bar{v}/b$, only the cyclotron motion itself introduces such high frequencies. When the high-frequency component is separated out by Taylor expanding the interaction field with respect to the cyclotron motion, one sees that the perpendicular velocity components are scattered by a dipole interaction. Of course, during the interaction, the z motion of the electrons is determined primarily by the monopole term.

Because the dipole interaction falls off with increasing particle separation as $1/r^3$ rather than $1/r^2$, small-impact parameter collisions are more important than large-impact parameter collisions. (Here, the quantity ρ serves as the impact parameter.) We will see that the term in the collision operator which describes the scattering of the perpendicular velocity components is constructed mainly from impact parameters of order b. The effect of such collisions may be described by a Boltzmann collision operator, since b is small compared to the interparticle spacing (i.e., $b\ll n^{-1/3}$).

Since the parallel velocity components can be scattered by a low-frequency field, the interaction responsible for this scattering falls off as $1/r^2$. We will see that the term in the collision operator which describes this scattering is proportional to an integral over impact parameters which diverges logarithmically at large impact parameter. We remove the divergence by imposing an *ad hoc* cutoff for impact parameters larger than λ_D. Of course, this procedure is not rigorous; it is to be understood in the sense of Landau's derivation of the Fokker–Planck collision operator (for an unmagnetized plasma) starting from the Boltzmann collision operator.[11]

In any case, we argue that the most important collisions are those characterized by small-impact parameter (i.e., $\rho\sim b$), and the Boltzmann operator provides a proper description of the effect of these collisions. Only these collisions can produce a scattering of both the perpendicular and the parallel velocity components. The collisions characterized by large-impact parameters (i.e., $\rho\gg b$) are trivial in the sense that they produce negligible change in the perpendicular components and at most an interchange of the parallel components (i.e., $v_{1z}\rightleftarrows v_{2z}$).

Because of the strong magnetic field, the Boltzmann operator may not be used in its usual form. We refer to the modified operator as a Boltzmann-like operator and derive the form of the operator from the BBGKY hierarchy following the arguments of Bogoliubov.[8] The one-particle distribution is governed by the equation

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\frac{\partial}{\partial\mathbf{r}_1}-\Omega\mathbf{v}_1\times\hat{z}\cdot\frac{\partial}{\partial\mathbf{v}_1}\right)f_1(1,t)=\frac{ne^2}{m}\int d\mathbf{r}_2\int d\mathbf{v}_2\,\frac{\partial}{\partial\mathbf{r}_1}\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\cdot\frac{\partial f_2}{\partial\mathbf{v}_1}(1,2,t), \tag{19}$$

and the two-particle distribution is governed by the equation

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{r}_2} - \Omega\mathbf{v}_1\times\hat{z}\cdot\frac{\partial}{\partial \mathbf{v}_1} - \Omega\mathbf{v}_2\times\hat{z}\cdot\frac{\partial}{\partial \mathbf{v}_2} - \frac{e^2}{m}\frac{\partial}{\partial \mathbf{r}_1}\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\cdot\left(\frac{\partial}{\partial \mathbf{v}_1}-\frac{\partial}{\partial \mathbf{v}_2}\right)\right]f_2(1,2,t)$$
$$= \frac{ne^2}{m}\int d\mathbf{r}_3\int d\mathbf{v}_3\left(\frac{\partial}{\partial \mathbf{r}_1}\frac{1}{|\mathbf{r}_1-\mathbf{r}_3|}\cdot\frac{\partial}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{r}_2}\frac{1}{|\mathbf{r}_2-\mathbf{r}_3|}\cdot\frac{\partial}{\partial \mathbf{v}_2}\right)f_3(1,2,3,t), \quad (20)$$

where $f_s(1,\ldots,s,t) = f_s(r_1,v_1,\ldots,r_s,v_s,t)$ is the s-particle distribution.

In accord with the previous arguments, we drop the three-particle term on the right-hand side Eq. (20), and later correct this omission by imposing a cutoff on the interaction for particle separations larger than λ_D. Of course, one can easily check that in the region of $(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2)$ space where small-impact parameter collisions take place (i.e., $|\mathbf{r}_1-\mathbf{r}_2|\sim b$), the right-hand side of Eq. (20) is negligible compared to the last term on the left-hand side. The time derivative in Eq. (20) may be dropped, since the two-particle distribution is assumed to have relaxed to become a functional of the one-particle distribution. This adiabatic assumption of Bogoliubov[8] requires that the duration of a collision be short compared to the inverse of the collision frequency, and this is the case here. Of course, a slow time dependence of f_2 remains, since f_2 follows f_1 adiabatically and f_1 undergoes collisional relaxation. Making these modifications in Eq. (22) and integrating over $(\mathbf{r}_2,\mathbf{v}_2)$ yields

$$\int d\mathbf{r}_2\int d\mathbf{v}_2\left(\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{r}_2} - \Omega\mathbf{v}_1\cdot\hat{z}\cdot\frac{\partial}{\partial \mathbf{v}_1} - \Omega\mathbf{v}_2\times\hat{z}\cdot\frac{\partial}{\partial \mathbf{v}_2}\right)f_2(1,2,t)$$
$$= \frac{e^2}{m}\int d\mathbf{r}_2\int d\mathbf{v}_2\frac{\partial}{\partial \mathbf{r}_1}\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\cdot\frac{\partial}{\partial \mathbf{v}_1}f_2(1,2,t), \quad (21)$$

where $\int d\mathbf{v}_2\,\partial f/\partial \mathbf{v}_2 = 0$ has been used. The right-hand side of this equation has the same form as the right-hand side of Eq. (19); so we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{r}_1} - \Omega\mathbf{v}_1\times\hat{z}\cdot\frac{\partial}{\partial \mathbf{v}_1}\right)f_1(1,t)$$
$$= n\int d\mathbf{r}_2\int d\mathbf{v}_2\left(\mathbf{v}_1\cdot\frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2\cdot\frac{\partial}{\partial \mathbf{r}_2} - \Omega\mathbf{v}_1\times\hat{z}\cdot\frac{\partial}{\partial \mathbf{v}_1} - \Omega\mathbf{v}_2\times\hat{z}\cdot\frac{\partial}{\partial \mathbf{v}_2}\right)f_1(1,2,t). \quad (22)$$

We assume that the plasma is homogeneous; so the distributions may be written as $f_1(1,t) = f_1(\mathbf{v}_1,t)$ and $f_2(1,2,t) = f_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r},t)$, where $r = \mathbf{r}_2 - \mathbf{r}_1$. Also, it it convenient to introduce the velocity variables $(u,w,\psi) = (v_z,|v_+|^2/2,\psi)$. In terms of these quantities, Eq. (22) takes the form

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \psi_1}\right)f_1(1,t)$$
$$= n\int d\mathbf{r}\int_{-\infty}^{+\infty} du_2\int_0^\infty dw_2\int_0^{2\pi} d\psi_2 \times\left[(\mathbf{v}_2-\mathbf{v}_1)\cdot\frac{\partial}{\partial \mathbf{r}} + \Omega\left(\frac{\partial}{\partial \psi_1}+\frac{\partial}{\partial \psi_2}\right)\right]f_2(1,2,t). \quad (23)$$

The right-hand side of this equation has the dimensions νf_1, where ν is an effective collision frequency. We look for a perturbation solution, treating ν and $(1/f_1)(\partial f_1/\partial t)$ as small compared to Ω. Substituting the expansion $f_1 = f_1^{(0)} + f_1^{(1)} + \cdots$ into Eq. (23) yields in lowest order $\Omega(\partial f_1^{(0)}/\partial\psi_1) = 0$ so that $f_1^{(0)} = f_1^{(0)}(u,w,t)$. In the next order, one obtains an equation involving $\Omega(\partial f_1^{(1)}/\partial\psi_1)$. This equation can be solved for an $f_1^{(1)}$ which satisfies the condition $f_1^{(1)}(\psi=0) = f_1^{(1)}(\psi=2\pi)$ only if the following constraint is satisfied:

$$2\pi\frac{\partial f_1^{(0)}}{\partial t} = n\int_0^{2\pi} d\psi_1\int_{-\infty}^{+\infty} du_2\int_0^\infty dw_2\int_0^{2\pi} d\psi_2\int d\mathbf{r} \times(\mathbf{v}_2-\mathbf{v}_1)\cdot\frac{\partial f_2}{\partial \mathbf{r}}, \quad (24)$$

where use has been made of $\int_0^{2\pi} d\psi_1\int_0^{2\pi} d\psi_2\,(\partial/\partial\psi_1 + \partial/\partial\psi_2)f_2 = 0$.

In regions of $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r})$ space which are such that the two electrons could not yet have interacted, we assume that the electrons are uncorrelated, that is, that $f_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r},t) = f_1(\mathbf{v}_1,t)f_1(\mathbf{v}_2,t)$. In the regions where the two electrons must already have interacted, the two-particle distribution is given by $f_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r},t) = f_1(\mathbf{v}_1',t)f_2(\mathbf{v}_2',t)$, where $(\mathbf{v}_1',\mathbf{v}_2')$ evolves into $(\mathbf{v}_1,\mathbf{v}_2)$ during the interaction. We have used here the fact that f_2 is constant along particle trajectories, which follows from setting the right-hand side of Eq. (20) equal to zero. Since the values of x and y change only slightly during an interaction (i.e., $\Delta x, \Delta y\sim r_L$) and since $(\mathbf{v}_1',\mathbf{v}_2')\simeq(\mathbf{v}_1,\mathbf{v}_2)$ for a large-impact parameter, we can set $f_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r},t) = f_1(\mathbf{v}_1,t)f_1(\mathbf{v}_2,t)$ in any region where x or y is large.

These results can be used to evaluate the $\mathbf{r}$ integral on the right-hand side of Eq. (24). In particular, we find that

$$\int_{-\infty}^{+\infty} dx\frac{\partial f_2}{\partial x} = \int_{-\infty}^{+\infty} dy\,\frac{\partial f_2}{\partial y} = 0, \quad (25)$$

and that

$$\int_{-\infty}^{+\infty} dz(v_{2z}-v_{1z})\frac{\partial f_2}{\partial z} = |v_{2z}-v_{1z}|\left[f_1(\mathbf{v}_1',t)f_1(\mathbf{v}_2',t) - f_1(\mathbf{v}_1,t)f_1(\mathbf{v}_2,t)\right]. \quad (26)$$

To lowest order in ν/Ω, one can replace $f_1(\mathbf{v},t)$ by $f_1^{(0)}(u,w,t)$ in the four distributions on the right-hand side of Eq. (26). Equation (24) then reduces to a Boltzmann-like collision operator

$$\frac{\partial f_1^{(0)}}{\partial t}(u_1,w_1,t) = n\int_0^\infty 2\pi\rho\,d\rho\int_{-\infty}^{+\infty} du_2\int_0^\infty dw_2\int_0^{2\pi} d\psi_2 \times|u_2-u_1|\left[f_1^{(0)}(u_1',w_1',t)f_1^{(0)}(u_2',w_2',t) - f_1^{(0)}(u_1,w_1,t)f_1^{(0)}(u_2,w_2,t)\right], \quad (27)$$

where we have used $\int_0^{2\pi} d\psi_1\int_0^{2\pi} d\psi_2(\quad) = 2\pi\int_0^{2\pi} d\psi_2(\quad)$ and have set $dx\,dy = 2\pi\rho\,d\rho$. The form of this collision operator could have been guessed at the outset, but it is satisfying to see it come out of the analysis.

IV. FOKKER–PLANCK COLLISION OPERATOR

In Sec. II, we found that the change in w_j (i.e., in $|v_{j+}|^2/2$) during a collision is small; so the distributions in the collision operator may be Taylor expanded. For example, $f_1^{(0)}(u_2',w_2')$ may be replaced by

$$f_1^{(0)}(u_2',w_2') = f_1^{(0)}(u_2',w_2) + \frac{\partial f_1^{(0)}}{\partial w_2}(u_2',w_2)(w_2' - w_2) + \frac{1}{2}\frac{\partial^2 f_1^{(0)}}{\partial w_2^2}(u_2',w_2)(w_2' - w_2)^2. \tag{28}$$

By recalling that primed variables refer to before the interaction (i.e., $t = t_a$) and unprimed variables to after the interaction (i.e., $t = t_b$), we see that Eq. (17) gives an expression for $w_2 - w_2'$ in terms of primed variables. To evaluate the right-hand side of Eq. (28), we want just the opposite, that is, $w_2' - w_2$ expressed in terms of unprimed variables. This can be obtained from Eq. (17) through the use of time reversal. By recalling that time reversal requires a reversal of the magnetic field (i.e., $B \to -B$ or $\delta\theta \to -\delta\theta$) we obtain

$$w_2' - w_2 = -\delta\theta\sqrt{w_2 w_1}\sin(\psi_2 - \psi_1) + [(\delta\theta)^2/4](w_1 - w_2). \tag{29}$$

The expansion for $f_1^{(0)}(u_1',w_1')$ is obtained by interchanging the subscripts 1 and 2 in these equations.

When the expansions are substituted into Eq. (27) and the ψ_2 integral is carried out, the terms that are first order in $\delta\theta$ vanish. Retaining the zero-order and second-order terms yields the collision operator

$$\frac{\partial f_1^{(0)}}{\partial t}(u_1,w_1,t) = n\int_0^\infty 2\pi\rho\, d\rho \int_{-\infty}^{+\infty} du_2 \int_0^\infty 2\pi\, dw_2 |u_2 - u_1| \Big\{ [f_1^{(0)}(u_1',w_1,t) f_1^{(0)}(u_2',w_2,t) - f_1^{(0)}(u_1,w_1,t) f_1^{(0)}(u_2,w_2,t)] + \frac{(\delta\theta)^2}{4} f_1^{(0)}(u_2',w_2,t)\frac{\partial}{\partial w_1}\Big(w_1 f_1^{(0)}(u_1',w_1,t) + w_2 w_1 \frac{\partial}{\partial w_1} f_1^{(0)}(u_1',w_1,t)\Big)\Big\}. \tag{30}$$

The first term in the curly brackets, that is the term which is zero order in $\delta\theta$, describes the effect on the distribution of the interchange of parallel velocities, and the second term describes the effect on the distribution of the change in w.

Let us check that the first term diverges logarithmically unless a large-impact parameter cutoff is introduced. Since interchange of parallel velocity components occurs (i.e., $u_2' = u_1$ and $u_1' = u_2$) when the maximum potential is large enough to reflect the electrons from one another, we can rewrite the first term in Eq. (30) as

$$\left(\frac{\partial f_1^{(0)}}{\partial t}(u_1,w_1,t)\right)_1 = n\int_{-\infty}^{+\infty} du_2 \int_0^\infty 2\pi\, dw_2 |u_2 - u_1| \pi\rho_0^2 \times [f_1^{(0)}(u_2,w_1,t) f_1^{(0)}(u_1,w_2,t) - f_1^{(0)}(u_1,w_1,t) f_1^{(0)}(u_2,w_2,t)], \tag{31}$$

where $\rho_0 = \rho_0(|u_2 - u_1|)$ is given by $m|u_2 - u_1|^2/2 = 2e^2/\rho_0$. Because ρ_0^2 varies as $|u_2 - u_1|^{-4}$, the u_2 integral receives its main contribution from values of u_2 near u_1. This means that the distribution functions may be Taylor expanded as

$$[f_1^{(0)}(u_2,w_1,t) f_1^{(0)}(u_1,w_2,t) - f_1^{(0)}(u_1,w_1,t) f_1^{(0)}(u_2,w_2,t)] \simeq (u_2 - u_1)\left(\frac{\partial f_1^{(0)}}{\partial u_1}(u_1,w_1,t) f_1^{(0)}(u_1,w_2,t) - f_1^{(0)}(u_1,w_1,t)\frac{\partial f_1^{(0)}}{\partial u_1}(u_1,w_2,t)\right) + \frac{(u_2 - u_1)^2}{2}\left(\frac{\partial^2 f_1^{(0)}}{\partial u_1^2}(u_1,w_1,t) f_1^{(0)}(u_1,w_2,t) - f_1^{(0)}(u_1,w_1,t)\frac{\partial^2 f^{(0)}}{\partial u_1^2}(u_1,w_2,t)\right). \tag{32}$$

When the expansion is substituted into Eq. (31), the first term drops out because it is odd in $(u_2 - u_1)$, and the second term yields

$$\left(\frac{\partial f_1^{(0)}}{\partial t}(u_1,w_1,t)\right)_1 = \frac{16\pi e^4}{m^2}\int_0^\infty \frac{d|u_2 - u_1|}{|u_2 - u_1|}\int_0^\infty 2\pi\, dw_2 \frac{\partial}{\partial u_1}\left(\frac{\partial f_1^{(0)}}{\partial u_1}(u_1,w_1,t) f_1^{(0)}(u_1,w_2,t) - f_1^{(0)}(u_1,w_1,t)\frac{\partial f_1^{(0)}}{\partial u_1}(u_1,w_2,t)\right). \tag{33}$$

By using $\int_0^\infty d|u_2 - u_1|/|u_2 - u_1| = \int_0^\infty d\rho_0/2\rho_0$, the logarithmic divergence at large-impact parameter is made evident. The divergence is removed by replacing the bare Coulomb interaction with a Debye screened interaction, which changes the defining equation for $\rho_0(|u_2 - u_1|)$ to $m|u_2 - u_1|^2/2 = 2(e^2/\rho_0)\exp(-\rho_0/\lambda_D)$. Also, there is an apparent divergence for a small-impact parameter, but this is simply an artifact of the failure of expansion (32) for $\rho \lesssim b$.

Introducing cutoffs at $\rho_0 = b$ and $\rho_0 = \lambda_D$ yields the result

$$\left(\frac{\partial f_1^{(0)}}{\partial t}(u_1,w_1,t)\right)_1 = \frac{n8\pi e^4}{m^2}\ln\left(\frac{\lambda_D}{b}\right)\int_0^\infty 2\pi\, dw_2 \times \frac{\partial}{\partial u_1}\left(\frac{\partial f_1^{(0)}}{\partial u_1}(u_1,w_1,t)f_1^{(0)}(u_1,w_2,t) - f_1^{(0)}(u_1,w_1,t)\frac{\partial f_1^{(0)}}{\partial u_1}(u_1,w_2,t)\right). \quad (34)$$

In Appendix B, Eq. (34) is compared to the term in Rostoker's collision operator which describes the scattering of the parallel velocity components.

Returning to a analysis of Eq. (30), we note that the first term may be characterized by an effective collision frequency of order $\nu_1 \sim \bar{v}nb^2 \ln(\lambda_D/b)$ and that the second term may be characterized by an effective collision frequency of order $\nu_2 \sim \bar{v}nb^2\langle(\delta\theta)^2\rangle \sim \bar{v}nb^2(r_L/b)^2$. Since $\nu_2 \ll \nu_1$, the electron distribution first relaxes in the manner required by the first term and then relaxes in the manner required by the second term. From the fact that the first term describes the interchange of parallel velocities, one can see that this term forces the distribution to relax to the form $f_1^{(0)}(u,w,t) = g(u)h(w,t)$, where

$$g(u) = \int_0^\infty 2\pi\, dw\, f_1^{(0)}(u,w,t), \qquad h(w,t) = \int_{-\infty}^{+\infty} du\, f_1^{(0)}(u,w,t). \quad (35)$$

By integrating Eq. (30) over dw_1, one can check that $\partial g/\partial t = 0$.

To investigate the effect of the second term in Eq. (30), we set $f_1^{(0)}(u,w,t) = g(u)h(w,t) + \delta f(u,w,t)$, where δf is presumed to be small. When this expression is substituted into Eq. (30), δf need be retained only in the first term of the collision operator. Integration over du_1 makes this term vanish, so Eq. (30) reduces to

$$\frac{\partial h}{\partial t}(w,t) = \nu_2\frac{\partial}{\partial w}\left(wh(w,t) + w\bar{w}\frac{\partial h}{\partial w}(w,t)\right), \quad (36)$$

where the subscript 1 on the w has been dropped, use has been made of the normalization

$$\int_0^\infty 2\pi\, dw\, h(w,t) = \int_0^\infty 2\pi\, dw \int_{-\infty}^{+\infty} du\, f_1^{(0)}(u,w,t) = 1, \quad (37)$$

and the following two quantities have been introduced

$$\bar{w} = \int_0^\infty 2\pi\, dw\, h(w,t)w, \quad (38)$$

$$\nu_2 = n\int_0^\infty 2\pi\rho\, d\rho \int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} du_1\, g(u_1)g(u_2) \times |u_1 - u_2|[\delta\theta(\rho,u_1-u_2)]^2. \quad (39)$$

As is expected, the time evolution implied by Eq. (36) is consistent with the definitions in Eqs. (37) and (38), that is, integrating Eq. (36) over dw yields

$$\frac{d}{dt}\int_0^\infty 2\pi\, dw\, h(w,t) = 0, \quad (40)$$

and integrating over $w\, dw$ yields

$$\frac{d}{dt}\int_0^\infty 2\pi\, dw\, h(w,t) = 0. \quad (41)$$

Also, we note that Eq. (36) takes a form that may be more obvious to some readers when written in terms of the variable $v_\perp = |v_+| = (2w)^{1/2}$. Rewriting the equation in terms of this variable yields

$$\frac{\partial h}{\partial t} = \frac{\nu_2}{2}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\left(v_\perp h + \frac{\overline{v_\perp^2}}{2}\frac{\partial h}{\partial n}\right), \quad (42)$$

where $(1/v_\perp)(\partial/\partial v_\perp)v_\perp$ is simply the divergence in cylindrical coordinates.

The expression for ν_2 may be simplified. Using the expression in Eq. (15) for $\delta\theta$ and changing the integral over ρ to an integral over α yields

$$\nu_2 = \frac{n}{\Omega^2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} du_1\, g(u_1)g(u_2)|u_1-u_2|^3 \times 2\pi\int_0^\infty \alpha\, d\alpha\, \eta^2(\alpha), \quad (43)$$

where

$$\eta(\alpha) = \int_{\sigma(\alpha)}^\infty d\xi \frac{[(\xi^2+1)^{-3/2} - \frac{3}{2}(\xi^2+1)^{-5/2}]}{[1-\alpha(\xi^2+1)^{-1/2}]^{1/2}}. \quad (44)$$

A numerical evaluation of the α integration yields

$$\nu_2 \simeq (2.6)\frac{n}{\Omega^2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} du_1\, g(u_1)g(u_2)|u_1-u_2|^3, \quad (45)$$

which is of order $\nu^2 \sim (n/\Omega^2)\bar{v}^3 \sim nb^2\bar{v}(r_L/b)^2$.

Equation (36) has an interesting property. As one expects, it is nonlinear in the sense that ν_2 and $\bar{w}$ are determined by the distribution function. However, ν_2 and $\bar{w}$ are time independent and are determined by the initial distribution. Thus, from the point of view of an initial value problem, the equation is effectively linear. One can easily verify that the solution is given by

$$h(w,t) = e^{-w/\bar{w}}\sum_{n=0}^\infty a_n L_n\left(\frac{w}{\bar{w}}\right)e^{-n\nu_2 t}, \quad (46)$$

where the $L_n(x)$ are Laguerre polynominals.[12] The constants a_n are given by

$$a_n = \int_0^\infty \frac{dw}{\bar{w}}L_n\left(\frac{w}{\bar{w}}\right)h(w,0). \quad (47)$$

For example, $L_0 = 1$, so $a_0 = (2\pi\bar{w})^{-1}$. Of course, this implies that $h(w,t=\infty) = \exp(-w/\bar{w})/(2\pi\bar{w})$.

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APPENDIX A: AN ALTERNATIVE DERIVATION OF THE ADIABATIC INVARIANT

In this appendix, we consider collisions involving the simultaneous interaction of many electrons. First, let us consider the question of energy exchange between the parallel and perpendicular velocity components.

To see that there is an adiabatic invariant which prevents such exchange even for a many electron collision, we introduce canonical coordinates[13] for each electron ($z, p_z, Y, m\Omega X, \psi, P_\psi$), where

$$\begin{aligned} \tan\psi &= -v_x/v_y, \\ P_\psi &= m(v_x^2+v_y^2)/2\Omega, \\ X &= x - v_y/\Omega, \quad Y = y + v_x/\Omega. \end{aligned} \tag{A1}$$

Here, ψ is the gyroangle and P_ψ is its conjugate momentum, and (X,Y) are the coordinates of the guiding center, $P_Y \equiv m\Omega X$ being the momentum conjugate to Y. The Hamiltonian for the electrons is given by

$$H = \sum_{j=1}^{N} \frac{p_{z_j}^2}{2m} + \Omega P_{\psi_j} + \sum_{i<j}^{N} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{A2}$$

where

$$\begin{aligned} |\mathbf{r}_i - \mathbf{r}_j|^2 = {} & (X_i + \rho_i\cos\psi_i - X_j - \rho_j\cos\psi_j)^2 \\ & + (Y_i + \rho_i\sin\psi_i - Y_j - \rho_j\sin\psi_j)^2 \\ & + (z_i - z_j)^2. \end{aligned} \tag{A3}$$

The quantity $\rho \equiv (2P_\psi/m\Omega)^{1/2}$ is the Larmor radius for an electron.

Assuming that the dynamics under discussion is that of a many-electron collision, rather than that of a collective mode (e.g., a plasma oscillation), the inequality $\Omega \gg v/b$ implies that the variables ψ_j are rapidly varying (i.e., $\dot\psi_j = \partial H/\partial P_{\psi_j} \sim \Omega$) compared to the other variables. Since there are many fast variables (i.e., ψ_j for $j = 1,\ldots,N$), the existence of an adiabatic invariant is not immediately obvious.

To uncover the invariant, we make a transformation to a new set of variables which is such that only one of the variables is rapidly varying. The transformation takes $\{(\psi_j, P_{\psi_j}) | j = 1,\ldots,N\}$ into $\{(\theta_j, P_{\theta_j}) | j = 1,\ldots,N\}$ via the generating function[14]

$$F_2 = P_{\theta_1}\psi_1 + \sum_{j=2}^{N} P_{\theta_j}(\psi_j - \psi_1), \tag{A4}$$

and leaves the variables $(z_j, p_{z_j}, Y_j, m\Omega X_j)$ unchanged. Of course, an identity transformation for these latter variables could have been added to the generating function. The new variables are related to the old by taking partial derivatives of the generating function in the usual manner[14]

$$\theta_1 = \frac{\partial F_2}{\partial P_{\theta_1}} = \psi_1, \quad \theta_j = \frac{\partial F_2}{\partial P_{\theta_j}} = \psi_j - \psi_1 \quad \text{for } j > 1, \tag{A5}$$

$$P_{\psi_1} = \frac{\partial F_2}{\partial \psi_1} = P_{\theta_1} - \sum_{j=2}^{N} P_{\theta_j}, \quad P_{\psi_j} = \frac{\partial F_2}{\partial \psi_j} = P_{\theta_j} \quad \text{for } j > 1. \tag{A6}$$

From Eq. (A6), it follows that $P_{\theta_1} = \Sigma_{j=1}^N P_{\psi_j}$; so the Hamiltonian takes the form

$$H = P_{\theta_1}\Omega + \sum_j \frac{p_{z_j}^2}{2m} + \sum_{i<j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{A7}$$

From the Hamiltonian, one can see that θ_1 is the only rapidly varying variable. Also, when the variation of the slow variables is suppressed, (θ_1, P_{θ_1}) are action angle variables. Thus, we may identify $P_{\theta_1} = \Sigma_{j=1}^N P_{\psi_j}$ as an adiabatic invariant. From the definition of P_{ψ_j} in Eq. (A1), we see that the invariant expresses conservation of the energy in the perpendicular velocity components.

Next, let us consider the parallel velocity components. For a binary interaction, conservation of momentum parallel to the field and conservation of parallel energy imply that the parallel velocity components for the two electrons can at most interchange. One can easily see that there is no such restriction for a three (or more) electron collision. Thus, many electron collisions should drive the distribution of parallel velocity components to a Maxwellian.

APPENDIX B: COMPARISON TO THE ROSTOKER COLLISION OPERATOR

In this Appendix, we compare the collision operator derived in this paper to that derived earlier by N. Rostoker.[1] He obtained the generalization for the case of a magnetized plasma of the Lenard–Balescu collision operator[15]

$$\begin{aligned} \frac{\partial f}{\partial t} = {} & \frac{\omega_p^4}{n} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{l,l'} \int d^3\mathbf{v}' \left(k_z \frac{\partial}{\partial v_z} + \frac{l\Omega}{v_\perp}\frac{\partial}{\partial v_\perp} \right) \frac{J_l^2(k_\perp v_\perp/\Omega) J_{l'}^2(k_\perp v_\perp'/\Omega)\, \pi\delta[k_z(v_z - v_z') + \Omega(l - l')]}{(k_z^2 + k_\perp^2)^2 |\epsilon(k_z v_z + l\Omega, \mathbf{k})|^2} \\ & \times \left[f(\mathbf{v}', t)\left(k_z \frac{\partial}{\partial v_z} + \frac{l\Omega}{v_\perp}\frac{\partial}{\partial v_\perp} \right) f(\mathbf{v}, t) - f(\mathbf{v}, t)\left(k_z \frac{\partial}{\partial v_z'} + \frac{l'\Omega}{\partial v_\perp'}\frac{\partial}{\partial v_\perp'} f(\mathbf{v}', t) \right) \right], \end{aligned} \tag{B1}$$

where J_l is the Bessel function of order l and

$$\epsilon(\omega, \mathbf{k}) = 1 + \sum_l \frac{\omega_p^2}{k^2} \int d\mathbf{v} \frac{J_l^2(k_\perp v_\perp/\Omega)[k_z(\partial/\partial v_z) + (l\Omega/v_\perp)(\partial/\partial v_\perp)] f}{k_z v_z + l\Omega - \omega} \tag{B2}$$

is the plasma dielectric function in the electrostatic approximation. The operator is derived with the aid of the weak interaction approximation, and an *ad hoc* cutoff is imposed for impact parameters which are smaller than or of the order of b, that is, the $\mathbf{k}$ integral is restricted to the domain $|\mathbf{k}| \lesssim 1/b$.

From the cutoff and from the argument of the delta function, one can see that in the regime $r_L \ll b$ only the $l = l'$ terms need be retained. Let us consider the terms with $l = l' \neq 0$; these terms involve the scattering of the perpendicular velocity components. Since $|\mathbf{k}| \lesssim 1/b \ll \Omega/v_\perp$, the Bessel functions may be replaced by small-argument expansions; so, near the cutoff, the k integral is of the form

$$\int d\mathbf{k}\,\frac{k_\perp^{4l}}{(k_z^2+k_\perp^2)^2}\sim\int^{1/b} dk_\perp\, k_\perp^{4l-2}\sim(1/b)^{4l-1}. \quad \text{(B3)}$$

The sensitivity of these terms to the *ad hoc* cutoff is simply a reflection of the fact that the most important collisions are characterized by small-impact parameters, and the weak interaction approximation fails there.

The term for $l=l'=0$ describes the scattering of the parallel velocity components. If we assume that the dielectric function simply imposes a cutoff for $|\mathbf{k}|\leqslant 1/\lambda_D$, this term reduces to

$$\left(\frac{\partial f(\mathbf{v},t)}{\partial t}\right)_{l=l'=0}=\frac{4\pi e^4 n}{m^2}\ln\left(\frac{\lambda_D}{b}\right)\int_0^\infty 2\pi v_\perp'\,dv_\perp' \times\frac{\partial}{\partial v_z}\left(f(v_z,v_\perp',t)\frac{\partial}{\partial v_z}f(v_z,v_\perp,t) - f(v_z,v_\perp,t)\frac{\partial f(v_z,v_\perp',t)}{\partial v_z}\right). \quad \text{(B4)}$$

This differs by a factor of 2 from the corresponding term derived in this paper [e.g., Eq. (34)]. Even though the scattering of parallel velocities is due mainly to collisions with impact parameters which are much larger than b, the weak interaction approximation is not valid. The parallel velocities remain unchanged unless the electrons reflect from one another, and integration along unperturbed orbits fails when reflection occurs.

Even though the collision operator is not rigorously applicable in the regime $r_L \ll b$, it does exhibit the adiabatic invariant. For the terms with $l=l'$, one can show that $\int d^3\mathbf{v}(mv_\perp^2/2)\partial f/\partial t=0$, that is, that the energy associated with the perpendicular velocity components is conserved.

[1]N. Rostoker, Phys. Fluids **3**, 922 (1960).
[2]D. Montogomery, G. Joyce, and L. Turner, Phys. Fluids **17**, 2201 (1974).
[3]W. B. Thompson, *Proceedings of the International School of Physics "Enrico Fermi," Course XXV, Advanced Plasma Theory* (Academic, New York, 1964), p. 1.
[4]P. P. J. M. Schram, Physica **45**, 165 (1969).
[5]M. J. Haggerty, Phys. Fluids **10**, 2199 (1967).
[6]K. Matsuda, Phys. Fluids **26**, 1508 (1983).
[7]A. H. Øien, J. Plasma Phys. **21**, 401 (1979).
[8]G. Uhlenbeck and G. Ford, *Lectures in Statistical Mechanics* (AMS, Providence, Rhode Island, 1963), p. 118.
[9]J. M. Malmberg (private communication).
[10]J. H. Malmberg and T. M. O'Neill, Phys. Rev. Lett. **39**, 1333 (1977).
[11]D. Montogomery and D. Tidman, *Plasma Kinetic Theory* (Mc Graw-Hill, New York, 1964), p.15.
[12]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965), p. 1039.
[13]G. R. Smith and A. N. Kaufmann, Phys. Fluids **21**, 2230 (1978).
[14]H. Goldstein, *Classical Mechanics* (Addison Wesley, New York, 1980), 2nd ed., p. 383.
[15]D. Montogomery and D. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964), p. 82.