Numerical study of a many-particle adiabatic invariant

P. G. Hjorth and T. M. O'Neil
Department of Physics, University of California, San Diego, La Jolla, California 92093

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For a pure electron plasma in a sufficiently strong magnetic field, an unusual many-electron adiabatic invariant is predicted to constrain the collisional dynamics. To the extent that this invariant is preserved, no exchange of energy is possible between the parallel and perpendicular degrees of freedom; the system can acquire and maintain two different temperatures, $T_{\parallel}$ and $T_{\perp}$. However, since an adiabatic invariant is not an exact constant of the motion, equilibration eventually takes place, but on an exponentially long time scale. This Letter presents the results of a molecular dynamics simulation, which verifies the existence of this unusual invariant and verifies the theoretically predicted equilibration rate.

This Letter presents the results of a molecular dynamics simulation of a strongly magnetized pure electron plasma. The simulation supports the recent theoretical prediction that the collisional dynamics of such a plasma is constrained by a many-electron adiabatic invariant.[1] Before describing the simulation, we review the basic elements of the theory.

We say that a pure electron plasma is strongly magnetized when the characteristic cyclotron radius $\bar{r}_c = \bar{v}/\Omega_c$ is small compared to the minimum separation between electrons. The theory has been developed for the case of a weakly correlated plasma, where the minimum separation is the classical distance of closest approach, $\bar{b} = 2e^2/T$. In this case, the condition that the plasma be strongly magnetized is the inequality $\bar{r}_c \ll \bar{b}$, which can be rewritten as $T^{3/2}$ (eV) $\ll 10^{-7}$ B (G); so one can see that strong magnetization requires low temperatures as well as large magnetic field. This unusual parameter regime is realized in a current series of experiments, where a magnetically confined pure electron plasma cools to the cryogenic temperature range as the electrons lose kinetic energy through cyclotron radiation.[2]

The existence of the adiabatic invariant is justified by an argument based on time scale considerations.[1] We start by considering the Hamiltonian for N electrons that move in a uniform external magnetic field and interact electrostatically. As canonical variables, it is convenient to decribe each electron in terms of its guiding-center variables plus its gyroangle ψ and conjugate momentum P_ψ. The gyroangle describes the cyclotron motion, and the conjugate momentum is the cyclotron action, $P_\psi = mv_\perp^2/2\Omega_c$, where $\mathbf{v}_\perp$ is the component of the electron velocity perpendicular to the magnetic field.

The gyroangles vary at the cyclotron frequency Ω_c and, as an estimate of the highest frequency associated with the guiding-center variables, we take the frequency characteristic of a close collision, $\bar{v}/\bar{b}$. The criterion for strong magnetization thus implies that the gyroangles are all rapidly varying compared to the guiding-center variables (i.e., $\Omega_c \gg \bar{v}/\bar{b}$). Because there are N fast variables, ψ_i $(i = 1,...,N)$, the existence of an adiabatic invariant is not yet apparent. To uncover the invariant, we make a canonical transformation that takes the variables ψ_i $(i = 1,...,N)$ to the variables χ_i $(i = 1,...,N)$, where $\chi_1 = \psi_1$ and $\chi_i = \psi_i - \psi_1$ for $i = 2,...,N$. In other words, all of the χ_i except χ_1 are relative variables and, thus, are slow variables. Since χ_1 is the only rapid variable, its conjugate momentum P_{χ_1} is an adiabatic invariant. This momentum turns out to be the total action associated with the cyclotron motion, $P_{\chi_1} = \Sigma m v_{\perp j}^2/2\Omega_c$.

This result has a simple physical interpretation. One may think of the original cyclotron variables as a collection of high-frequency oscillators and of the guiding-center variables as a collection of low-frequency oscillators. The high-frequency oscillators can exchange quanta with each other, but not with the low-frequency oscillators; so the total action associated with the high-frequency oscillators is very nearly conserved.

For the case of a uniform magnetic field, one may equivalently say that the total perpendicular kinetic energy, $\Sigma m v_{\perp j}^2/2$, is an adiabatic invariant, and this invariant constrains the collisional relaxation of an anisotropic velocity distribution. The usual time scale for such a relaxation is the time scale for a few collisions. On this time scale, the adiabatic invariant is well conserved, and there is negligible exchange of energy between the parallel and the perpendicular degrees of freedom. Thus, the distribution of parallel and the distribution of perpendicular velocities become Maxwellian separately, with the parallel temperature ($T_{\parallel}$) not necessarily equal to the perpendicular temperature ($T_\perp$).

The evolution does not stop at this stage, since an adiabatic invariant is not strictly conserved; it suffers exponentially small changes. In the present case, each collision produces an exponentially small exchange of energy between the parallel and the perpendicular degrees of freedom, and these act cumulatively in such a way that the two distributions acquire a common temperature. However, the rate of relaxation is exponentially small.

The simple case of an isolated binary collision can be treated analytically, and we find that the small exchange of parallel and perpendicular energy is of the form

$$\Delta E_\perp = (16)^{-1} m^2 v_\perp v_\parallel^2 (\rho/e^2) \times \cos(\delta) h(\epsilon,\rho/b) \exp[-g(\rho/b)/\epsilon], \qquad (1)$$

where the velocities are relative velocities (i.e., $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$), ρ is the distance between the field lines on which the electrons move, and δ is an initial gyroangle. The quantity ϵ is the small parameter $\epsilon = v_\parallel/\Omega_c b$, where $b = e^2/(\mu v_\parallel^2/2)$ and $\mu = m/2$ is the reduced mass. The function $h(\epsilon,\rho/b)$ is nonexponential, and $g(\rho/b)$ is a monotonically increasing func-

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tion with $g(0) = \pi/2$ and $g(\rho/b) \approx \rho/b$ for ρ/b large. As one would expect, the argument of the exponential is of the form $\Omega_c \tau$, where τ is the duration of the binary collision. For a head-on collision (i.e., $\rho = 0$), $\tau \approx \pi b/2v_\parallel$ and for a large impact parameter collision ($\rho/b \gg 1$), $\tau \approx \rho/v_\parallel$.

Clearly, the collisions that are most effective in producing an exchange of parallel and perpendicular energy are those characterized by small impact parameters (i.e., $\rho \lesssim b$). For a weakly correlated plasma, such collisions are well separated binary collisions (i.e., $\rho \lesssim \bar{b} \ll n^{-1/3}$); so their effect on the velocity distribution can be treated with a Boltzmann-like collision operator.[1,3] From such a treatment, we find the rate equation[1]

$$\frac{dT_\perp}{dt} = \frac{(T_\parallel - T_\perp)}{T_\parallel T_\perp} \frac{n}{4} \int_0^\infty 2\pi\rho \, d\rho \times \int d\mathbf{v}\, v_\parallel f_r(v_\parallel, v_\perp)(\Delta E_\perp)^2 , \qquad (2)$$

where

$$f_r(v_\parallel, v_\perp) = \left(\frac{\mu}{2\pi T_\parallel}\right)^{1/2} \left(\frac{\mu}{2\pi T_\perp}\right) \exp\left(-\frac{\mu v_\parallel^2}{2T_\parallel} - \frac{\mu v_\perp^2}{2T_\perp}\right) \qquad (3)$$

is the distribution of relative velocities. It is the square of $(\Delta E_\perp)$ that is averaged over impact parameter and relative velocity, because the factor $\cos(\delta)$ in Eq. (1) varies randomly from collision to collision.

Equation (2) can be rewritten as

$$\frac{dT_\perp}{dt} = (T_\parallel - T_\perp) n \bar{b}^2 \bar{v}_\parallel I(\bar{\epsilon}) , \qquad (4)$$

where $\bar{b}$ and $\bar{\epsilon}$ are simply b and ϵ with $v_\parallel$ replaced by $\bar{v}_\parallel = \sqrt{T_\parallel/\mu}$. The factor $n\bar{b}^2\bar{v}_\parallel$ is very nearly the collision frequency, or equilibration rate, for an unmagnetized plasma (i.e., $\nu_{ee} = \sqrt{2\pi}\ln(\Lambda) n\bar{b}^2\bar{v}_\parallel$), and the factor $I(\bar{\epsilon})$ is the reduction in this rate for the case of strong magnetization. For small $\bar{\epsilon}$, this reduction is given by

$$I(\bar{\epsilon}) \simeq (0.47)(\bar{\epsilon})^{1/5} \exp[-(2.05)/(\bar{\epsilon})^{2/5}] . \qquad (5)$$

Notice that the function $I(\bar{\epsilon})$ varies exponentially as $(\bar{\epsilon})^{-2/5}$ rather than as $(\bar{\epsilon})^{-1}$. This dependence arises through a competition between two effects. Since ϵ varies as $v_\parallel^3$, the quantity $(\Delta E_\perp)^2 \sim \exp(-\pi/\epsilon)$ is a rapidly increasing function of $v_\parallel$. On the other hand, $f_r(v_\parallel, v_\perp)$ is a rapidly decreasing function of $v_\parallel$; so the integral in Eq. (2) contains the product of two functions that compete at large $v_\parallel$. Physically this is easy to understand: very few collisions involve large relative velocities, but the collisions that do involve large relative velocities are very effective in producing an exchange of parallel and perpendicular energy. A saddle-point evaluation of the integral yields the small $\bar{\epsilon}$ asymptotic expression $I(\bar{\epsilon})$ given in Eq. (5).

We now describe the numerical simulation. For computational ease, we scale velocities by $\bar{v}_\parallel = \sqrt{T_\parallel/\mu}$, distances by $\bar{b} = 2e^2/\mu\bar{v}_\parallel^2$, and times by $\bar{b}/\bar{v}_\parallel$. With these units, the equations of motion take the form

$$\dot{\mathbf{x}}_i' = \mathbf{v}_i' , \quad \dot{\mathbf{v}}_i' = \frac{1}{\bar{\epsilon}} \mathbf{v}_i' x \hat{z} + \frac{1}{4} \sum_{i \neq j}^{N} \frac{\mathbf{x}_i' - \mathbf{x}_j'}{|\mathbf{x}_i' - \mathbf{x}_j'|^3} , \quad i = 1,\ldots,N , \qquad (6)$$

where primes signify scaled variables. Because we need to evaluate the full Coulomb force term, the number of floating point operations associated with each reference to this set of equations scales roughly quadratically with the number N of particles in the system; this is the main obstacle to a simulation involving a realistically large number of particles. Nevertheless, a glimpse of the phenomena has been obtained with a simulation involving 50 interacting particles, performed on the SDSC CRAY X-MP.

As initial conditions, we take the 50 particles to be uniformly distributed spatially inside a box $-L/2 < (x,y,z) < +L/2$, of volume L^3, and with initial velocities picked from a bi-Maxwellian velocity distribution with $(T_\perp/T_\parallel) = 0.2$. As the system evolves, the particles are confined in the z direction by specular reflections at the walls at $z = \pm L/2$. Confinement in the radial direction is ensured, since the total canonical angular momentum P_θ is a constant of the motion and provides a constraint on the allowed radial positions of the electrons.[4] Another constant of the motion is the total energy; we employ a high precision Bulirsch–Stoer ODE solver[5] and find during a typical run that the total energy is conserved to order $\Delta E/E \sim 10^{-7}$ and that the total canonical angular momentum is conserved to order $\Delta P_\theta/P_\theta \sim 10^{-5}$.

From a statistical (or macroscopic) perspective, the scaled system is characterized by two parameters: $\bar{\epsilon}$ and L'. Here, we have in mind fixed $N = 50$, fixed initial $T_\perp/T_\parallel = 0.2$, and fixed initial $\langle v_\parallel'^2 \rangle = (T_\parallel/m)/(T_\parallel/\mu) = \frac{1}{2}$. The parameter $\bar{\epsilon}$ is a measure of the magnetic field strength, and the parameter L' is a measure of density $n' = N/L'^3$. By decreasing L', we increase the collision frequency (ν_{ee}), or equivalently, we increase the correlation strength ($\Gamma = e^2/aT_\parallel$, where $4\pi n a^3/3 = 1$). Not all of the $(\bar{\epsilon}, \Gamma)$ parameter plane is physically accessible. For a nonneutral electron plasma, the Brillouin limit[6] (i.e., $\omega_p < \Omega_e/\sqrt{2}$) specifies the maximum density that can be confined for a given magnetic field strength, and in terms of $\bar{\epsilon}$ and Γ this limit takes the form $\bar{\epsilon}\Gamma^{3/2} < 1/\sqrt{12}$.

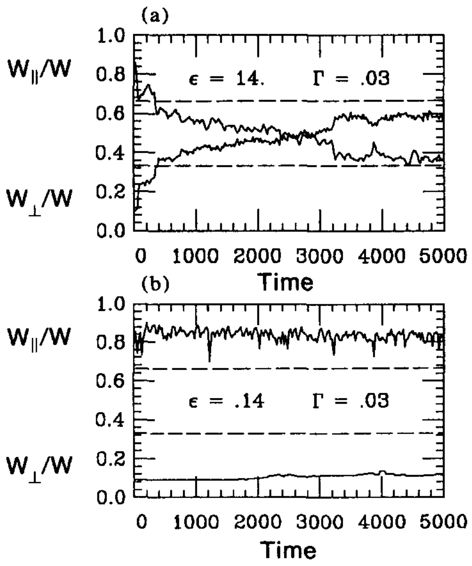

FIG. 1. Evolution of $W_\perp/W$ and $W_\parallel/W$ (a) for weak magnetic field (i.e., $\bar{\epsilon} = 14$) and (b) for strong magnetic field (i.e., $\bar{\epsilon} = 0.14$).

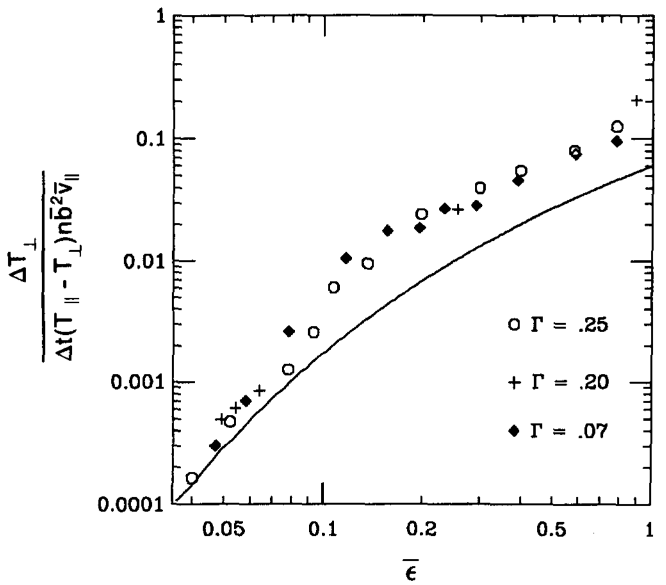

FIG. 2. Comparison of the function $I(\bar{\epsilon})$ (solid curve) to values of $(\Delta T_\perp/\Delta t)[(T_\parallel - T_\perp)n\bar{b}^2\bar{v}_\parallel]^{-1}$ obtained from the simulations.

Some sample results are shown in Figs. 1(a) and 1(b), where $W_\perp/W$ and $W_\parallel/W$ are plotted as functions of time. Here, $W_\perp$ and $W_\parallel$ refer to the total perpendicular and parallel kinetic energies, respectively, and $W = W_\perp + W_\parallel$. The dashed lines mark the values predicted by the equipartition theorem: $W_\perp/W = \frac{2}{3}$ and $W_\parallel/W = \frac{1}{3}$. Figure 1(a) shows the results of a run with low magnetic field strength ($\bar{\epsilon} = 14$) and Fig. 1(b) shows the results of a run with increased field strength ($\bar{\epsilon} = 0.14$) but the same correlation strength (i.e., $\Gamma = 0.03$). One can see that the increased field suppresses the relaxation. Also, one can check that the relaxation rate in Fig. 1(a) has the expected value, that is, $\nu_{ee} \simeq 8\ln(\bar{\epsilon})\Gamma^3$, where we have set $\Lambda \simeq \bar{\epsilon}$ since $\bar{r}_c \ll \lambda_D$.

To investigate the rate equation numerically [i.e., Eq. (4)], we examined the initial rate of temperature equilibrium for a bi-Maxwellian velocity distribution with $T_\perp = 0.2T_\parallel$. In order to suppress statistical fluctuations without going to a large number of particles, we averaged the change in perpendicular and parallel kinetic energy over 20 different sets of initial conditions. These were advanced forward a time Δt short compared with the equipartition time, but long enough for a least-squares fit to the slope of the evolving $W_\perp(t) = NT_\perp(t)$. In Fig. 2, the analytic prediction $(dT_\perp/dt)[(T_\parallel - T_\perp)n\bar{b}^2\bar{v}_\parallel]^{-1} = I(\bar{\epsilon})$ is compared to numerical values of $(\Delta T_\perp/\Delta t)[(T_\parallel - T_\perp)n\bar{b}^2\bar{v}_\parallel]^{-1}$ for various values of $\bar{\epsilon}$ and Γ. One can see that the numerical values are insensitive to the value of Γ and follow the $I(\bar{\epsilon})$ curve quite well, although the agreement is best in the small $\bar{\epsilon}$ limit. This is to be expected, since $I(\bar{\epsilon})$ was obtained in the small $\bar{\epsilon}$ asymptotic limit.

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