Stability theorem for off-axis states of a non-neutral plasma column

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A sufficient condition is given for the stability of a long non-neutral plasma column that obeys two-dimensional **E**×**B** dynamics. The column is confined by a uniform magnetic field and bounded by a conducting cylinder aligned with the field. The variational approach used here generalizes the well-known stability of a centered, axisymmetric column, whose density is a monotonically decreasing function of radius. Displacement of such a column away from the axis by excitation of an $l=1$ diocotron mode yields a dynamical equilibrium stationary in a frame rotating with the mode. This new equilibrium is shown to be stable if the column is not too large. The analysis may explain, in part, the remarkable longevity observed for $l=1$ diocotron modes in experiments.

I. INTRODUCTION

Consider a long non-neutral plasma column that is confined by a uniform magnetic field, **B**, in a region of space that is bounded by a conducting cylindrical wall. Suppose that the plasma column is centered on the axis of the cylinder (which is parallel to the magnetic field), is cylindrically symmetrical, and has a density profile that is a monotonically decreasing function of radius. As has been known for many years, such a column is stable under two-dimensional (2-D) **E**×**B** drift perturbations.[1]

When a diocotron mode with azimuthal harmonic number $l=1$ is excited on such a column, the column is effectively displaced off the axis of the cylinder and then rotates about the axis at the frequency of the diocotron mode.[2,3] We may consider this state to be an off-axis dynamical equilibrium, since the plasma is stationary in a frame that rotates with the mode frequency. In this paper, we show that this off-axis equilibrium is also stable to all two-dimensional (2-D) **E**×**B** drift perturbations, provided that the radius of the column is sufficiently small. Our stability analysis may explain, in part, the remarkable longevity observed experimentally for $l=1$ diocotron modes (more than 10^5 cycles).[4]

The discussion is carried out with a sign convention corresponding to a plasma of negative charges (e.g., a pure electron plasma), but the stability results are valid for non-neutral plasmas with either sign of charge. For a negatively charged plasma, the **E**×**B** drift rotation of the column and the diocotron mode rotation are in a positive sense (relative to **B**), and it is easier to talk about (and think about) positive frequencies than negative frequencies.

The 2-D **E**×**B** drift dynamics is governed by the continuity equation,

$$\frac{\partial n}{\partial t}+\frac{c}{B}\hat{z}\times\nabla\phi\cdot\nabla n=0, \tag{1}$$

and Poisson's equation,

$$\nabla^2\phi=4\pi en. \tag{2}$$

Here, $\phi(r,\theta)$ is the electric potential, $n(r,\theta)$ is the density, and a minus sign has been introduced explicitly on the right-hand side of Poisson's equation ($e>0$). Also, we have introduced a cylindrical coordinate system (r,θ,z), dropped any dependence on z, and made use of the fact that **E**×**B** drift flow in a uniform **B** field is incompressible [i.e., $\nabla\cdot(\hat{z}\times\nabla\phi)=0$]. The electric potential must be constant, say zero, on the conducting wall [i.e., $\phi(r=R,\theta)=0$]. These equations also describe the 2-D incompressible and inviscid flow of a neutral fluid, where $(c/B)\phi$ is the streamfunction and $-4\pi en(c/B)$ is the z component of the vorticity. However, one should note that the boundary condition $\phi(R,\theta)=0$ corresponds to a slip condition at the wall for the case of a neutral fluid.

Here 2-D **E**×**B** drift dynamics conserves the electrostatic energy,

$$W=-\int\frac{ne\phi}{2}\,d^2\mathbf{r}, \tag{3}$$

and the canonical angular momentum,

$$P_\theta=-\int\frac{e}{c}A_\theta(r)rn\,d^2\mathbf{r}, \tag{4}$$

where the θ component of the vector potential is given by $A_\theta(r)=Br/2$ for a uniform axial magnetic field. It is useful to introduce the quantity $\overline{W}=W-\omega P_\theta$, which is the electrostatic energy in a frame that rotates with frequency ω; this quantity also is conserved.

The logic of the stability argument is to show that the functional $\overline{W}$ is a maximum for a particular state, $n(r,\theta)$, as compared to all other states that are accessible under incompressible flow. Since $\overline{W}$ is conserved and the **E**×**B** drift flow is incompressible, no further change in state is possible, that is, the particular state is a stable equilibrium that is stationary in the rotating frame ω. If there is a degeneracy so that the maximum value of $\overline{W}$ is shared by many contiguous states, then the plasma may occupy any one of these states. For the off-axis equilibria, there is such a degeneracy associated with the azimuthal symmetry of the geometry; the stability argument fixes the shape and radial location of the off-axis column but not the azimuthal location. (Of course, the rotation frequency ω is fixed.)

There are many examples of this kind of stability theorem in the literature, and the review article by Holm *et al.*

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has an extensive bibliography.[5] From the reasoning in the previous paragraph it is clear that stability is implied whenever $\overline{W}$ is either a maximum or a minimum, and for most examples in the literature the criterion for $\overline{W}$ to be a minimum is established. Formally, it is much easier to establish this criterion than the criterion that $\overline{W}$ be a maximum. Nevertheless, both cases were discussed in pioneering work by Kelvin and Arnold.[6] For the case at hand, we will see that the criterion that $\overline{W}$ be a minimum cannot be satisfied by the off-axis states, although there is a suggestion to the contrary in the literature.[7] Consequently, for these states we are forced to undertake the more difficult task of establishing the criterion that $\overline{W}$ be a maximum. (We are concerned here with what Holm *et al.* call *formal* stability. We do not obtain bounds on perturbation norms—i.e., *nonlinear* stability—and do not know whether the distinction has any practical importance for **E**$\times$**B** dynamics.)

This paper can be thought of as an extension of our recent work on the statistical mechanics of a system of long charged rods (or point vortices) confined in cylindrical geometry.[8] For certain values of the total electrostatic energy and canonical angular momentum, the maximum entropy state is an off-axis dynamical equilibrium of the kind considered here. We know that this state is stable to all 2-D **E**$\times$**B** drift perturbations (the kind of dynamics underlying the statistical mechanics), since it is a state of maximum entropy. However, the density profile is of a special form—the thermal equilibrium form. Here, we argue that a plasma column of sufficiently small radius and *any* monotonically decreasing density profile is stable when displaced off axis. The situation is similar to that for velocity space instabilities. One knows from statistical mechanics that a homogeneous plasma with a Maxwellian velocity distribution is stable, but one can argue, more generally, that any distribution that is a monotonically decreasing function of kinetic energy is stable under Vlasov dynamics (incompressible flow in phase space).[9]

Section II contains a stability proof for the case where the radius of the column is very small but the displacement of the column off the axis is not necessarily small. This proof has a simple physical interpretation. Section III contains a more formal analysis based on variational theory and bifurcation theory. This analysis does not assume that the column radius is small, but is limited to small displacements off the axis. In general, when a column is displaced off the axis it rotates about the axis with a frequency that is shifted slightly from the diocotron mode frequency, as given by linear theory. One may think of this as the nonlinear frequency shift of the diocotron mode. The analysis in Sec. II shows that the off-axis column is stable when the nonlinear frequency shift is positive. Of course, in all this discussion we assume that the density profile for the column is a monotonically decreasing function of radius before the column is shifted off the axis. In Sec. IV, we argue that the nonlinear frequency shift is positive for a column of sufficiently small radius (i.e., $r_p \lesssim R/\sqrt{2}$). Section V contains a discussion of the results and some speculations about further implications of the stability theorem. In particular, we argue that cylindrical symmetry of the confinement apparatus is not necessary for stable confinement within the context of 2-D **E**$\times$**B** drift dynamics.

II. PLASMA COLUMN OF SMALL RADIUS

It is instructive to start with the case of a column with very small radius, since the analysis is easy to interpret physically. In general, the electric potential can be written as $\phi=\phi_f+\phi_i$, where

$$\phi_f(\mathbf{r})=+\int 2en(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'|\,d^2\mathbf{r}', \tag{5}$$

is the free-space potential that is produced by the plasma column in the absence of any conductors, and

$$\phi_i(\mathbf{r})=-\int 2en(\mathbf{r}')\ln\left|\left(\mathbf{r}-\mathbf{r}'\frac{R^2}{|\mathbf{r}'|^2}\right)\frac{|\mathbf{r}'|}{R}\right|d^2\mathbf{r}', \tag{6}$$

is the potential due to the charges on the conducting cylinder. In a frame that rotates with frequency ω, the electrostatic energy is given by the two terms

$$\overline{W}=-\int\frac{ne\phi_f}{2}\,d^2\mathbf{r}+\int n\left[-\frac{e\phi_i}{2}+\frac{\omega eBr^2}{2c}\right]d^2\mathbf{r}. \tag{7}$$

For $\overline{W}$ to be a maximum, it is necessary to maximize each integral separately. The first integral depends on the plasma shape, but is not changed by a shift in the plasma location. For a very small column, the second integral depends on the plasma location, but is insensitive to the plasma shape. To understand this, note that the bracket in the second integral varies only slightly over the region occupied by a small column. Thus, maximizing the first integral determines the shape of the column, and maximizing the second integral determines the location of the column.

The first integral is a maximum, relative to other states that are accessible under incompressible flow, provided that the column is cylindrically symmetrical about an axis through its center of mass and has a density profile that decreases monotonically outward from this axis. For a small cylindrical column that is displaced from the axis by a distance D, the value of the second integral is approximately

$$\overline{W}_2=e^2N^2\ln\left[\left(1-\frac{D^2}{R^2}\right)R\right]+\frac{eNB\omega D^2}{2c}, \tag{8}$$

where $N=\int d^2\mathbf{r}\,n(\mathbf{r})$. This expression is an extremum when

$$0=\frac{d\overline{W}_2}{dD}=\left(\frac{eNB}{c}\right)D\left(\frac{-\omega_d}{1-D^2/R^2}+\omega\right), \tag{9}$$

where we have introduced the frequency of the $l=1$ diocotron mode,[2,3] $\omega_d=(2eNc/BR^2)$. For $\omega<\omega_d$, there is only one root ($D=0$), and at this root

$$\frac{d^2\overline{W}_2}{dD^2}=\frac{eNB}{c}\left(\frac{-\omega_d}{1-D^2/R^2}+\omega\right)<0. \tag{10}$$

Thus, $\overline{W}_2$ is a maximum, and we may conclude that the on-axis state ($D=0$) is a stable equilibrium.

For $\omega > \omega_d$, there are two roots. At the $D=0$ root, $\overline{W}_2$ is not a maximum, since the large parenthesis in Eq. (10) is positive. At the second root [i.e., $\omega_d/(1-D^2/R^2)=\omega$], $\overline{W}_2$ is maximum since

$$\frac{d^2\overline{W}_2}{dD^2}=\left(\frac{eNB}{c}\right)D\left(\frac{-\omega_d 2D/R^2}{(1-D^2/R^2)^2}\right) \tag{11}$$

is negative. This root corresponds to a stable off-axis equilibrium that is stationary in a frame that rotates with the frequency $\omega=\omega_d/(1-D^2/R^2)$. Since $\overline{W}_2$ is independent of azimuth, the azimuthal location of the column is not fixed by the stability argument.

Note that one may not conclude that the on-axis state is unstable from the fact that $\overline{W}_2$ is not a maximum at $D=0$ for $\omega > \omega_d$. This negative result has no implication for either stability or instability. In fact, we know from the analysis for $\omega < \omega_d$ that the on-axis state is stable.

From Eq. (9) one can see that the rotation frequency, $\omega=\omega_d/(1-D^2/R^2)$, is determined by a balance between an inward force due to rotation in the magnetic field and an outward force due to the image charges. The nonlinear frequency shift,

$$\omega-\omega_d \sim \frac{\omega_d D^2/R^2}{1-D^2/R^2}, \tag{12}$$

is positive, since the force due to the image charges increases with increasing D faster than the first power of D. It turns out that for a column of relatively large radius, displacement of the column off the axis gives rise to a substantial quadrupole distortion of the column.[10,11] The quadrupole interaction with the image charges is repulsive and can change the sign of the frequency shift to be negative. By formally making the frequency shift in Eq. (9) negative, say, by substituting $\omega_d/(1+D^2/R^2)$ for $\omega_d/(1-D^2/R^2)$, one can check that $\overline{W}_2$ is no longer a maximum at the off-axis root.

III. VARIATIONAL AND BIFURCATION ANALYSIS

Let us consider a state that is characterized by density $n(r,\theta)$ and potential $\phi(r,\theta)$. If the density and potential undergo the variation $\delta n(r,\theta)$ and $\delta\phi(r,\theta)$, where $\nabla^2\delta\phi=+4\pi e\,\delta n$ and $\delta\phi(R,\theta)=0$, the variation in the functional $\overline{W}$ is given by

$$\delta\overline{W}=\int \bar{h}\,\delta n\,d^2\mathbf{r}-\frac{1}{2}\int e\,\delta\phi\,\delta n\,d^2\mathbf{r}, \tag{13}$$

where $\bar{h}=-e\phi+\omega eBr^2/2c$ is the potential energy of a charge $(-e)$ in a frame that rotates with frequency ω. We use the symbol $\bar{h}$ since the potential energy is the Hamiltonian for a test charge within the context of $\mathbf{E}\times\mathbf{B}$ dynamics. The variation in density is assumed to occur through incompressible flow, so the functional (generalized entropy),

$$S=\int G(n)d^2\mathbf{r}, \tag{14}$$

has zero variation $(\delta S=0)$.[5-7] Here, $G(n)$ is an arbitrary function. To second order, we may set

$$0=\delta S=\int G'(n)\delta n\,d^2\mathbf{r}+\frac{1}{2}\int G''(n)(\delta n)^2\,d^2\mathbf{r}. \tag{15}$$

Subtracting this equation from Eq. (13) then yields the result

$$\delta\overline{W}=\int[\bar{h}-G'(n)]\delta n\,d^2\mathbf{r}-\frac{1}{2}\int[e\,\delta\phi\,\delta n+G''(n)\times(\delta n)^2]d^2\mathbf{r}. \tag{16}$$

If the initial state is an equilibrium, the first-order variation (first term) can be made to vanish through a proper choice of $G(n)$. By transforming Eq. (1) to a frame that rotates with frequency ω, one can see that a state is stationary if the density is constant along equipotentials in the rotating frame, since these are the trajectories of guiding centers. This occurs if the density can be expressed as $n=n(\bar{h})$. Such a functional relationship implies that there exists a function $G(n)$ such that $[\bar{h}-G'(n)]=0$, and this implies that the first term vanishes for all δn. We assume that the initial state is such an equilibrium, so Eq. (16) reduces to the form

$$\delta\overline{W}=-\frac{1}{2}\int\left(e\,\delta\phi\,\delta n+\frac{d\bar{h}}{dn}(\delta n)^2\right)d^2\mathbf{r}, \tag{17}$$

where use has been made of $G''(n)=d\bar{h}/dn$.

Before proceeding, it is useful to enumerate the equilibria of interest. The simplest is an on-axis cylindrically symmetric state with a density profile, $n_0(r)$, that we assume is a monotonically decreasing function of r. Let $\phi_0(r)$ be the corresponding self-consistent potential and define $\bar{h}_0=-e\phi_0+\omega eBr^2/2c$. By eliminating r between $\bar{h}_0(r)$ and $n_0(r)$, one obtains a relationship of the form $n=n(\bar{h})$. As one would expect, this state is stationary in any rotating frame.

We approach the off-axis states as perturbations away from the on-axis state. We set

$$\begin{aligned} n&=n_0(r)+\Delta n(r,\theta),\\ \phi&=\phi_0(r)+\Delta\phi(r,\theta). \end{aligned} \tag{18}$$

Since the condition for a state to be an equilibrium is that the density be expressible as $n=n(\bar{h})$, and since the frequency shift must be second order (it is even in Δn), we have $\Delta n=(dn_0/d\bar{h}_0)(-e\Delta\phi)$ and

$$\frac{dn_0}{d\bar{h}_0}=\frac{dn_0}{dr}\left(\frac{d\bar{h}_0}{dr}\right)^{-1}=\frac{dn_0/dr}{(eBr/c)[\omega-\omega_E(r)]}. \tag{19}$$

(We use Δn to denote a change from one equilibrium to another, whereas δn denotes a variation about an equilibrium.) Here, $\omega_E(r)=(c/Br)(d\phi_0/dr)$ is the local $\mathbf{E}\times\mathbf{B}$ drift rotation frequency of the unperturbed column. The perturbed density and potential must satisfy Poisson's equation,

$$\nabla^2(\Delta\phi)+\frac{4\pi ec}{Br}\frac{dn_0}{dr}\frac{(\Delta\phi)}{[\omega-\omega_E(r)]}=0, \tag{20}$$

subject to the boundary condition $\Delta\phi(R,\theta)=0$.

This is the eigenfunction equation for an $l=1$ diocotron mode, and for $\omega=\omega_d$ the solution is known to be

$$\Delta\phi=D\cdot(Br/c)[\omega_E(r)-\omega_d]\cos(\theta-\delta),$$
$$\Delta n=D\cdot\frac{dn_0}{dr}\cos(\theta-\delta), \tag{21}$$

where D and δ are constants.[2,3] The boundary condition on $\Delta\phi$ is satisfied since $\omega_d=\omega_E(R)$. From the expression for Δn, it follows that the equilibrium can be interpreted as a displacement D of the column off the axis.

The analysis here is to first order in D, and to this order the equilibrium is stationary in a frame that rotates with frequency $\omega=\omega_d$. In the next section, we carry the analysis to higher order and obtain the nonlinear frequency shift $\omega-\omega_d=(D/R)^2A$, where the constant A depends on $n_0(r)$ and R. At each order of the perturbation theory, care is taken to make sure that the off-axis state is accessible from the on-axis state through incompressible flow. This is clearly the case for the linear expressions in Eq. (21), since the new state is obtained by displacing the column. In higher order, there is distortion of the column as well as displacement. The parameter D is then defined to be the displacement of the column center of mass. From an experimental (or operational) perspective, the off-axis state is reached through incompressible flow, if the $l=1$ diocotron mode is excited adiabatically.

Figure 1(a) shows the locus of equilibria in the (ω,D) plane for the case of a positive frequency shift $(A>0)$. The line at $D=0$ represents the on-axis equilibria and the parabola through $\omega=\omega_d$ represents the off-axis equilibria. From the figure, one can see why this case is called a forward pitchfork bifurcation.[12] The case of negative frequency shift $(A<0)$ is shown in Fig. 1(b); this case is called a backward pitchfork bifurcation. Finally, the arbitrary phase factor δ in Eq. (21) means that there are infinitely many off-axis equilibria for a given value of D. Of course, this is a consequence of the azimuthal symmetry. To represent all of these equilibria we must regard D as a two-dimensional vector and rotate the parabolas in Figs. 1(a) and 1(b) about the ω axis.

To determine the stability of a particular equilibrium $n=n(\bar{h})$, we must examine the second variation of $\bar{W}$, as given by Eq. (17). Stability against small-amplitude perturbations is implied when $\bar{W}$ is either a local maximum ($\delta\bar{W}<0$ for all allowed δn) or a local minimum ($\delta\bar{W}>0$ for all allowed δn). It is relatively easy to establish the criterion that $\bar{W}$ be a local minimum.[1,5,7] By using $\nabla^2\,\delta\phi=4\pi e\,\delta n$ and $\delta\phi(R,\theta)=0$, Eq. (17) can be rewritten as

$$\delta\bar{W}=\frac{1}{2}\int\left(\frac{(\nabla\delta\phi)^2}{4\pi}-\frac{d\bar{h}}{dn}(\delta n)^2\right)d^2\mathbf{r}, \tag{22}$$

so $\delta\bar{W}>0$ for all δn, provided that $d\bar{h}/dn<0$.

This criterion was used to show that a cylindrically symmetric equilibrium with a monotonically decreasing density profile, $n=n_0(r)$, is locally stable.[1,7] Such an equilibrium is stationary in any rotating frame [and, correspondingly, a relationship of the form $n=n_0(\bar{h}_0)$ exists for any rotating frame], so there is flexibility in the choice of frame where the analysis is to be carried out. In a rapidly rotating frame [i.e., $\omega>\omega_E(0)\geqslant\omega_E(r)$], it follows from Eq. (19) that $dn_0/d\bar{h}_0<0$, so $\bar{W}$ is a local minimum. Here, use has been made of the fact that $\omega_E(r)$ decreases monotonically when $n_0(r)$ does.

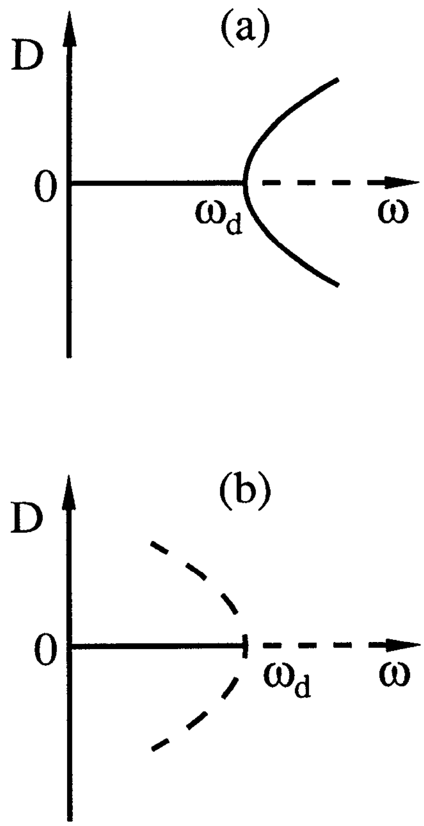

FIG. 1. Bifurcation diagrams for displacement D versus rotation frequency ω for the reference frame. (a) Small column, (b) large column. Solid curves indicate equilibria that are local maxima of $\bar{W}$; dashed curves, those that are saddle points of $\bar{W}$.

This stability argument cannot be extended to the off-axis equilibria, because the criterion $dn/d\bar{h}<0$ cannot be satisfied by these equilibria. Let us assume that there exists an off-axis equilibrium $n_1=f(\bar{h}_1)$ where $df/d\bar{h}_1<0$, and then prove a contradiction. Because the confinement geometry has cylindrical symmetry, the equilibrium state can be rotated through an arbitrary azimuthal angle to obtain another equilibrium $n_2=f(\bar{h}_2)$. From Poisson's equation in the rotating frame, $\nabla^2\bar{h}=-4\pi e^2 f(\bar{h})+2\omega eB/c$, and the boundary condition $\bar{h}(R,\theta)=\omega eBR^2/2c$, an integration by parts yields

$$\int d^2\mathbf{r}\,|\nabla(\bar{h}_1-\bar{h}_2)|^2=4\pi e^2\int d^2\mathbf{r}(h_1-h_2)[f(h_1)-f(h_2)]. \tag{23}$$

The left-hand side is positive and the right-hand side is negative (for $df/d\bar{h}<0$, by the mean value theorem), so there cannot exist an off-axis equilibrium $n=f(\bar{h})$ such that $df/d\bar{h}<0$. More explicitly, the reason that $df/d\bar{h}$ cannot be negative for an off-axis equilibrium is that the frequency ω of the frame in which the equilibrium is stationary is too low. For ω equal to or near the diocotron mode frequency [i.e., $\omega \approx \omega_d = \omega_E(R) \leqslant \omega_E(r)$], it follows from Eq. (19) that $dn_0/dh_0 > 0$.

For this situation, the two terms in the bracket of integral (22) have opposite signs. To prove stability, we are forced to undertake the relatively difficult task of showing that one of the two terms (the second, as it turns out) is larger than the other. When this is the case, $\delta \overline{W} < 0$ for all allowed δn, and $\overline{W}$ is a local maximum.

Before tackling the off-axis equilibria, it is useful to reexamine the on-axis equilibrium $n = n_0(r)$ [where $n_0'(r) < 0$], but this time in a slowly rotating frame. We will find that $\overline{W}$ is a local maximum when $\omega < \omega_d = \omega_E(R)$. To rewrite the second variation [Eq. (17)] in a more convenient form, it is useful to consider the eigenfunction problem

$$\psi_j + \frac{d\overline{h}}{dn}\frac{\nabla^2 \psi_j}{4\pi e^2} + \lambda_j \psi_j = 0, \tag{24}$$

where $\psi_j(R,\theta) = 0$. From this equation, one can derive the orthogonality condition

$$0 = (\lambda_j - \lambda_k) \int \nabla \psi_j \cdot \nabla \psi_k \, d^2\mathbf{r}.$$

As usual, degenerate eigenfunctions must be made orthogonal explicitly. It is convenient to normalize the eigenfunctions so that

$$\int \nabla \psi_j \cdot \nabla \psi_k \, d^2\mathbf{r} = 8\pi \delta_{jk} . \tag{25}$$

We assume that the set of eigenfunctions is complete and expand the perturbed potential and density as

$$\delta\phi = \sum_j a_j \psi_j ,$$

$$\delta n = \sum_j \frac{a_j(\nabla^2 \psi_j)}{(4\pi e)} . \tag{26}$$

Substituting into Eq. (17) and using Eqs. (24) and (25) yields the result

$$\delta \overline{W} = -\sum_j \lambda_j a_j^2. \tag{27}$$

If the eigenvalues are all positive, $\overline{W}$ is a local maximum. From Eq. (24) and the fact that $d\overline{h}/dn > 0$ (for $\omega \sim \omega_d$), it follows that the higher eigenvalues are large and positive. Also, by a happy circumstance, we can determine the value of ω where the lowest eigenvalue, λ_1, passes through zero. For ω on one side or the other of this critical value, all the eigenvalues must be positive.

By using Eq. (19), Eq. (24) can be rewritten as:

$$\nabla^2 \psi_j + (1+\lambda_j) \frac{4\pi ec}{Br} \frac{dn_0}{dr} \frac{\psi_j}{[\omega - \omega_E(r)]} = 0. \tag{28}$$

For $\lambda = 0$ and $\omega = \omega_d$, this reduces to the eigenfunction equation for the $l=1$ diocotron mode, so a solution is given by

$$\psi_1(r,\theta) = d \cdot (Br/c) [\omega_E(r) - \omega_d] \cos\theta, \tag{29}$$

where d is a constant. The boundary condition $\psi_1(R,\theta) = 0$ is satisfied since $\omega_d = \omega_E(R)$. The corresponding density perturbation,

$$\frac{+\nabla^2 \psi_1}{4\pi e} = d \cdot \frac{dn_0}{dr} \cos\theta, \tag{30}$$

results from a displacement, d, of the column off the axis. Here, d is determined by the normalization condition [Eq. (25)]

$$1 = a^2 \frac{eB}{4c} \int a^2\mathbf{r} \frac{dn_0}{dr} r[\omega_d - \omega_E(r)]. \tag{31}$$

Because of azimuthal symmetry, there exists a second eigenfunction,

$$\psi_2(r,\theta) = d \cdot (Br/c) [\omega_d - \omega_E(r)] \sin\theta, \tag{32}$$

that is degenerate with the first (i.e., $\lambda_1 = \lambda_2 = 0$). Of course, we have chosen the azimuthal phases so that the two eigenfunctions are orthogonal.

Since ψ_1 and ψ_2 have azimuthal mode number $l=1$ and only a single peak in their radial dependence, $\lambda_1 = \lambda_2$ are the lowest eigenvalues that enter expansion (27). In this regard, note that the expansion contains no $l=0$ eigenfunctions. As a first-order perturbation that is produced through incompressible flow, δn can be written as $\delta n = \hat{z} \times \nabla f \cdot \nabla n_0$, where $f = f(r,\theta)$. By using $\nabla n_0 = \hat{r}\, dn_0/dr$, this reduces to $\delta n = -(1/r)(\partial f/\partial\theta)(dn_0/dr)$, so the theta average of δn vanishes, and no $l=0$ terms can appear in expansion (27).

There remains the question of whether $\lambda_1(\omega) = \lambda_2(\omega)$ are positive for $\omega - \omega_d > 0$ or for $\omega - \omega_d < 0$. Treating $\omega - \omega_d$ as a small parameter and applying first-order perturbation theory to Eq. (28) yields the integral expression

$$\lambda_1(\omega) = + \int d^2\mathbf{r} \left(\frac{\nabla^2 \psi_1}{4\pi e}\right)^2 \frac{eBr}{2c} (\omega - \omega_d) \left(\frac{dr}{dn_0}\right). \tag{33}$$

Substituting from Eqs. (30) and (31) and carrying out the resulting integral, then yields the result

$$\lambda_1(\omega) = (\omega_d - \omega) C, \tag{34}$$

where

$$C = +2N \left(\int_0^R 2\pi r \, dr \frac{dn_0}{dr} [\omega_d - \omega_E(r)] \right)^{-1} \tag{35}$$

is positive. Thus, $\lambda_1(\omega)$ and $\lambda_2(\omega)$ are positive for $\omega < \omega_d$.

Finally, since $\lambda_1(\omega) = \lambda_2(\omega)$ are the two lowest eigenvalues that appear in expansion (27), all of the eigenvalues are positive for $\omega < \omega_d$. Thus, $\delta \overline{W}$ is negative for all δn (accessible under incompressible flow), $\overline{W}$ is a local maximum at the on-axis equilibrium, and we may conclude that the equilibrium is stable to small amplitude $\mathbf{E} \times \mathbf{B}$ drift perturbations.

For $\omega > \omega_d$, $\lambda_1(\omega) = \lambda_2(\omega)$ are both negative, but the remaining eigenvalues are positive. Thus $\overline{W}$ is a saddle at the on-axis equilibrium, and no conclusion concerning stability can be reached (by working in this frame).

Next we examine the extremal properties of $\overline{W}$ at an off-axis equilibrium that is close to the bifurcation point. The rotation frequency ω differs only slightly from ω_d, and the displacement off-axis $D_{\rm eq}(\omega)=R[(\omega-\omega_d)/A]^{1/2}$ is small. Let ψ_j' and λ_j' be the eigenfunctions and eigenvalues obtained from Eq. (25) with the function $d\overline{h}/dn$ given by the off-axis equilibrium. Since this equilibrium differs only slightly from the on-axis equilibrium, one expects by continuity that ψ_j' and λ_j' differ only slightly from ψ_j and λ_j. In particular, we know from Sturm–Liouville theory that eigenvalues are ordered by l and by the number of nodes. Thus λ_j' for $j\geqslant 3$ are positive and well separated from zero for ω close enough to ω_d. However, λ_1' and λ_2' are small (but may be positive, negative, or zero).

We determine λ_2' by appeal to the cylindrical symmetry of the confinement geometry. In general, the potential for an off-axis equilibrium can be written as $\phi=\phi(r,\theta-\delta)$, where δ is an arbitrary phase. When this potential is expressed as an expansion in $D_{\rm eq}(\omega)/R$, the first-order term is given by Eq. (21). Because of the cylindrical symmetry the function $n(\overline{h})$, which appears in Poisson's equation [i.e., $\nabla^2\phi=4\pi en(\overline{h})$], depends on δ only through ϕ. By taking the derivative of Poisson's equation with respect to δ, one sees that $d\phi/d\delta$ is a $\lambda=0$ solution of eigenfunction equation (22). Also, $d\phi/d\delta$ satisfies the boundary condition $d\phi/d\delta(R,\theta-\delta)=0$. To be specific, we choose the phase $\delta=0$, so at first order in $D_{\rm eq}(\omega)/R$, $d\phi/d\delta|_{\delta=0}$ is proportional to $\psi_2(r,\theta)$. Thus, we identify $d\phi/d\delta|_{\delta=0}$ as the eigenfunction $\psi_2'=0$ and conclude that $\lambda_2'=0$.

To determine λ_1' we develop a construction in the spirit of bifurcation theory.[12] Referring to expansion (27), we imagine that the plasma starts in the on-axis state and then is displaced a small amount in the direction of eigenfunction ψ_1. The new potential is $\phi_0(r)+[D/d]\psi_1(r,\theta)$. This process is continued with many infinitesimal steps, each of which occurs through incompressible flow and produces an increment δD, where D is the displacement of the column center of mass. At each stage the potential and density are related self-consistently through Poisson's equation, and the potential is determined as an expansion in D/R, with the first term being $[D/d]\psi(r,\theta)$. In this expansion there are no extra terms with coefficients that can be varied independently of D. Eventually, the plasma state passes through the off-axis equilibrium at $D=D_{\rm eq}(\omega)$. The off-axis equilibrium is, by definition, reached through such a sequence of infinitesimal flows. In this way we define a path through state space, starting from the on-axis equilibrium and passing through the off-axis equilibrium. By calculating the electrostatic energy along the path we obtain the function $\overline{W}=\overline{W}(\omega,D)$.

The path through state space passes through the on-axis equilibrium in the direction of eigenfunction ψ_1 and through the off-axis equilibrium in the direction of the eigenfunction ψ_1'. In fact, each step along the path is in the direction of the local ψ_1. Displacements in the direction of other eigenfunctions (ψ_j for $j\neq 1$) would correspond to extra terms in the expansion with amplitudes that can be varied independently of D, and these terms are, by definition, ruled out. Near the on-axis equilibrium, the expansion coefficient a_1 [see Eq. (27)] is given by $a_1=D/d$, and near the off-axis equilibrium, the expansion coefficient a_1' is given by $a_1'=[D-D_{\rm eq}(\omega)]/d'$. Thus, it follows that

$$\lambda_1=\frac{d^2}{2}\frac{\partial^2\overline{W}}{\partial D^2}\bigg|_{D=0},\qquad \lambda_1'=\frac{d'^2}{2}\frac{\partial^2\overline{W}}{\partial D^2}\bigg|_{D=D_{\rm eq}(\omega)}. \tag{36}$$

To relate λ_1 and λ_1', we construct a Taylor expansion of $\partial\overline{W}/\partial D$ that is valid in the region of the bifurcation. Since the first variation of $\overline{W}$ vanishes at an equilibrium, $\partial\overline{W}/\partial D$ must vanish at $D=0$, $D=D_{\rm eq}(\omega)$, and $D=-D_{\rm eq}(\omega)$. These are the three equilibria that are encountered by drawing a line at constant ω (for $\omega>\omega_d$) in Fig. 1(a) and (for $\omega<\omega_d$) in Fig. 1(b). Thus, to third order in D, we can set

$$\frac{\partial\overline{W}}{\partial D}=g(\omega)D[D^2-D_{\rm eq}^2(\omega)]. \tag{37}$$

Substituting into Eq. (36), and eliminating $g(\omega)$ between the resulting two equations, then yields the result

$$\lambda_1'(\omega)=-2(d'/d)^2\lambda_1(\omega)\sim-2\lambda_1(\omega), \tag{38}$$

where d'/d has been approximated by unity to lowest order in $D_{\rm eq}(\omega)/R$.

For the case where the nonlinear frequency shift is positive, the off-axis equilibrium is stationary in a frame with $\omega>\omega_d$. In this frame $\lambda_1(\omega)<0$, so $\lambda_1'(\omega)>0$. Thus, $\lambda_j'>0$ for all j except $j=2$ and $\lambda_2'=0$. From expansion (27) it follows that $\delta\overline{W}$ is negative for variations along any of the eigenfunctions, except that for $j=2$, and that $\delta\overline{W}=0$ for variations along this eigenfunction. As compared to other states accessible under incompressible flow, the off-axis equilibrium is a local maximum of $\overline{W}$, except for the degeneracy associated with the azimuthal symmetry. Thus, except for changes in the azimuthal orientation, the off-axis equilibrium is stable to small-amplitude $\mathbf{E}\times\mathbf{B}$ drift perturbations.

For the case where the nonlinear frequency shift is negative, the off-axis equilibrium is stationary in a frame with $\omega<\omega_d$. In this frame $\lambda_1(\omega)>0$ so $\lambda_1'(\omega)<0$. Thus, $\overline{W}$ is a saddle at the off-axis equilibrium, and no conclusion concerning stability can be made.

Referring again to Figs. 1(a) and 1(b), the solid lines represent equilibria, where $\overline{W}$ is a local maximum, and the dashed lines, where $\overline{W}$ is a local saddle. For the forward pitchfork bifurcation (positive frequency shift) in Fig. 1(a), the property of being a maximum of $\overline{W}$ is transferred at the bifurcation point from the on-axis equilibria to the off-axis equilibria. Such a transfer is typically the case for a forward pitchfork bifurcation, but not for a backward pitchfork bifurcation.[12] Note that Fig. 1(b) does not show such a transfer.

IV. NONLINEAR FREQUENCY SHIFT

In the previous section we found that the off-axis equilibria are stable to small-amplitude $\mathbf{E}\times\mathbf{B}$ drift perturbations, provided that the nonlinear frequency shift is positive. In this section, we show that this quantity is indeed positive for a column of sufficiently small radius.

Explicit calculations of the frequency shift have been done for the simple case of a waterbag model.[10,11] In this model, the density is uniform within some boundary curve, and is zero elsewhere, so the off-axis equilibrium clearly is accessible through incompressible flow from the on-axis equilibrium. When the center of mass of the column is displaced off the axis by a small but finite amount D, the nonlinear frequency shift is given by $\omega-\omega_d=A(D/R)^2$, where

$$A=\omega_d\left[\frac{1-2(r_p/R)^2}{[1-(r_p/R)^2]^2}\right]. \tag{39}$$

Here, r_p is the radius of the column in the on-axis (cylindrically symmetric) equilibrium. One can see that the frequency shift is positive when $r_p<R/\sqrt{2}$. This result is in reasonably good agreement with experiment, even when the plasma edge is rounded somewhat, that is, is not precisely modeled by the waterbag model.[4]

For a column of small radius but arbitrary monotonically decreasing density profile, the off-axis equilibrium (and frequency shift) can be obtained as an expansion in the small parameters r_p/R and D/R. Here, r_p is the characteristic radius and $\mathbf{D}$ is the displacement of the center of mass. Making the change of coordinates, $\boldsymbol{\rho}=\mathbf{r}-\mathbf{D}$ and $\boldsymbol{\rho}'=\mathbf{r}'-\mathbf{D}$ in Eq. (6) and Taylor expanding the logarithm with respect to ρ/R, $\rho'/R\approx r_p/R$, and D/R yields the result

$$\phi_i(\rho,\alpha)\sim\text{const}+\frac{2D\rho\cos(\alpha)}{R^2}\int d^2\rho'\;n(\rho',\alpha')\times\left(1+\frac{D^2}{R^2}\frac{\rho'^2}{R^2}\cos(2\alpha')\right)+\frac{D^2\rho^2\cos(2\alpha)}{R^4}\int d^2\rho'\;n(\rho',\alpha')+\cdots, \tag{40}$$

where α is the angle between $\boldsymbol{\rho}$ and $\mathbf{D}$ and α' is the angle between $\boldsymbol{\rho}'$ and $\mathbf{D}$. By construction, $n(\rho',\alpha')$ is independent of α' to lowest order in D/R; it is also consistent to assume that it is even in α'; these properties were used in deriving Eq. (40).

Substituting into the Hamiltonian

$$\bar{h}=-e(\phi_f+\phi_i)+\omega eB(\mathbf{D}+\boldsymbol{\rho})^2/2c, \tag{41}$$

yields the expression

$$\bar{h}\sim\text{const}-e\phi_f(\rho,\alpha)+\frac{\omega eB\rho^2}{2c}+\rho D\cos(\alpha)\,\frac{eB}{c}\left[\omega-\omega_d\left(1+\frac{D^2}{R^2}+\int\frac{d^2\rho'\;n(\rho',\alpha')\rho'^2\cos(2\alpha')}{NR^2}\right)\right]-\frac{e^2ND^2\rho^2\cos(2\alpha)}{R^4}. \tag{42}$$

The free space potential ϕ_f can be written as

$$\phi_f(\rho,\alpha)=\phi_0(\rho)+\delta\phi_f(\rho,\alpha)+\text{const}, \tag{43}$$

where $\phi_0(\rho)$ is the potential for the on-axis equilibrium, but shifted to the off-axis center of coordinates at $\mathbf{r}=\mathbf{D}$, and $\delta\phi_f(\rho,\alpha)\to 0$ as $D\to 0$. By introducing $h_0(\rho)=\phi_0(\rho)+\omega eB\rho^2/2c$ and by dropping constant terms, the Hamiltonian reduces to

$$\bar{h}\sim\bar{h}_0(\rho)+\rho D\cos(\alpha)\frac{eB}{c}\left[\omega-\omega_d\left(1+\frac{D^2}{R^2}+\int d^2\rho'\,\frac{n(\rho',\alpha')\rho'^2\cos(2\alpha')}{NR^2}\right)\right]-\frac{e^2ND^2\rho^2\cos(2\alpha)}{R^4}-e\delta\phi_f(\rho,\alpha). \tag{44}$$

For the off-axis state to be an equilibrium, the density must be expressible as $n=n(\bar{h})$. Also, we require that the off-axis state be accessible from the on-axis state through incompressible flow. It turns out that both of these conditions can be satisfied, if we set

$$n=n_0(\bar{h})\sim n_0(\bar{h}_0)+\frac{dn_0}{d\bar{h}_0}(\bar{h}-\bar{h}_0), \tag{45}$$

and then choose ω so that the square bracket in Eq. (44) vanishes, that is, so that

$$\bar{h}-\bar{h}_0\sim\frac{-e^2ND^2\rho^2\cos(2\alpha)}{R^4}-e\delta\phi_f(\rho,\alpha). \tag{46}$$

The remaining unknown, $\delta\phi_f(\rho,\alpha)$, must be determined self-consistently from Eq. (5), that is, from the integral equation

$$\delta\phi_f(\rho,\alpha)=\int d^2\rho'\;2e\,\delta n(\rho',\alpha')\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'|, \tag{47}$$

where $\delta n=(dn_0/d\bar{h}_0)(\bar{h}-\bar{h}_0)$. Noting that the $l=1$ contribution vanishes in the center-of-mass coordinate system, and using the expansion[13]

$$\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'|=\ln(\rho_>)-\sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{\rho_<}{\rho_>}\right)^l\cos[l(\alpha-\alpha')], \tag{48}$$

where $\rho_>=\max(\rho,\rho')$ and $\rho_<=\min(\rho,\rho')$, one can see that $\delta\phi_f(\rho,\alpha)=\delta\phi_f(\rho)\cos(2\alpha)$ and that $\delta\phi_f(\rho)\to 1/\rho^2$ as $\rho\to\infty$. This latter condition serves as a boundary condition for the differential equation

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{4}{\rho^2}\right)\delta\phi_f(\rho)+4\pi e^2\frac{dn_0}{d\bar{h}_0}\delta\phi_f(\rho)=\frac{-4\pi e^3ND^2\rho^2}{R^4}\frac{dn_0}{d\bar{h}_0}, \tag{49}$$

which is obtained by operating on both sides of Eq. (47) with the Laplacian. For simplicity, we solve the differential equation rather than the integral equation.

To this end, consider the eigenfunction equation

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{4}{\rho^2}\right)\psi_j+4\pi e^2\frac{dn_0}{d\bar{h}_0}(1+\lambda_j)\psi_j=0, \tag{50}$$

subject to the boundary condition $\psi_j(\rho)\to 1/\rho^2$ as $\rho\to\infty$. From this equation, one obtains the orthogonality condition

$$0=(\lambda_j-\lambda_k)\int_0^\infty \rho\, d\rho\,\frac{dn_0}{d\bar{h}_0}\psi_j\psi_k. \tag{51}$$

We normalize the eigenfunctions so that

$$\int_0^\infty \rho\, d\rho\,\frac{dn_0}{d\bar{h}_0}\psi_j\psi_k=\delta_{jk}. \tag{52}$$

Substituting the expansion

$$\delta\phi_f(\rho)=\sum_j a_j\psi_j(\rho) \tag{53}$$

into Eq. (49), and using Eqs. (50) and (52), yields an expression for the expansion coefficients,

$$a_j=\frac{+eND^2}{\lambda_j R^4}\int_0^\infty d\rho\,\rho^3\frac{dn_0}{d\bar{h}_0}\psi_j. \tag{54}$$

Thus we obtain the result

$$\delta\phi_f(\rho,\alpha)=\frac{+eND^2\cos(2\alpha)}{R^4}\sum_{j=1}^\infty \psi_j(\rho)\frac{1}{\lambda_j}\cdot\int_0^\infty d\rho'\,\rho'^3\frac{dn_0}{d\bar{h}_0}\psi_j(\rho'). \tag{55}$$

Operating on both sides of this equation with the Laplacian and using eigenfunction equation (50) yields the density perturbation

$$\delta n(\rho,\alpha)=\frac{-e^2ND^2}{R^4}\frac{dn_0}{d\bar{h}_0}\cos(2\alpha)\sum_{j=1}^\infty\psi_j(\rho)\times\left(1+\frac{1}{\lambda_j}\right)\int_0^\infty d\rho'\,\rho'^3\left(\frac{dn_0}{d\bar{h}_0}\right)\psi_j(\rho'). \tag{56}$$

Equivalently, this quantity could have been calculated from Eq. (45). At this stage one can see that the off-axis equilibrium is accessible through incompressible flow from the on-axis equilibrium. The density $n_0(r)$ is changed into $n_0(\rho)$ through a displacement of the column as a whole, and the density $n_0(\rho)$ is changed into $n_0(\rho)+\delta n(\rho,\alpha)$ through a quadrupole distortion. The distortion can be achieved through incompressible flow since $\delta n(\rho,\alpha)$ can be written as $\hat{z}\times\nabla f\cdot\nabla n$.

Finally, setting the square bracket in Eq. (44) equal to zero, and substituting for $n(\rho',\alpha')$ yields the nonlinear frequency shift $\omega-\omega_d=A(D/R)^2$, where

$$A=\omega_d\left[1-e^2\sum_{j=1}^\infty\left(1+\frac{1}{\lambda_j}\right)\pi\left(\int_0^\infty d\rho\,\frac{\rho^3}{R^2}\frac{dn_0}{d\bar{h}_0}\psi_j(\rho)\right)^2\right]. \tag{57}$$

The second term in the square bracket is of order $(r_p/R)^4$ and is negligible for sufficiently small r_p. Note that a Taylor expansion of the right-hand side of Eq. (39) yields $A\sim\omega_d[1-(r_p/R)^4]$.

V. DISCUSSION

We have considered the stability of a pure electron plasma column within the context of 2-D **E**×**B** drift dynamics. The column is confined by a uniform axial magnetic field in a region of space that is bounded by a cylindrical conducting wall. Previously it was known that an on-axis, cylindrically symmetric, plasma column with a monotonically decreasing density profile is locally stable.[1] We showed that this stability persists when the column is displaced off the axis, provided that the column radius is not too large. Displacement of the column is affected by exciting an $l=1$ diocotron mode, and our stability analysis may explain, in part, the remarkable longevity observed experimentally for these modes.

Some other dynamical equilibria, such as $l=2$ diocotron modes or multiple-vortex configurations, are saddle points of $\bar{W}$ because certain displacements decrease the energy functional. (The equilibrium frame rotation frequency $\omega>\omega_d$.) Hence our analysis does not solve the stability problem in these cases.

The stability analysis may also be relevant to recent experimental results on confinement in an apparatus without cylindrical symmetry.[14] In these experiments the cylindrical wall is divided azimuthally into several electrically isolated sectors. When the various sectors are maintained at different potentials, the wall ceases to be an equipotential surface, and the cylindrical symmetry of the geometry is destroyed. Nevertheless, long-lived (seconds) equilibria are observed experimentally.[14] At first glance this result is surprising; in general, it has been thought that cylindrical symmetry of the apparatus is required for good confinement.[15]

In its most general form, the stability argument developed here also requires cylindrical symmetry of the apparatus. We used conservation of the total canonical angular momentum P_θ and of the total electrostatic energy W to argue that $\bar{W}=W+\omega P_\theta$ is conserved. However, for the special case $\omega=0$ (the laboratory frame), $\bar{W}$ reduces to W, and we need not require that P_θ be conserved—cylindrical symmetry of the apparatus is not required. If the electrostatic energy W is a local maximum in the laboratory frame ($\omega=0$) for the equilibrium of interest, then that equilibrium is stable to small-amplitude **E**×**B** drift perturbations.

In general, the equilibria for an asymmetric trap are stationary only in the laboratory frame. Let $n=n(r,\theta)$ and $\phi=\phi(r,\theta)$ be the density and self-consistent potential for such an equilibrium. It is necessary that there exist a functional relationship $n=n(h)$, where $h=-e\phi$, and we require that this relationship be such that $dn/dh>0$. We can show that W is a local maximum for such an equilibrium in the limit where the asymmetries are small, that is, where the potentials on the various sectors are all close in value. In this case, the asymmetric equilibrium differs only slightly from a cylindrically symmetric equilibrium $n=n_0(h_0)$, where $dn_0/dh_0>0$ (or, equivalently, $dn_0/dr<0$). Of course, this latter equilibrium is realized when all of the sectors are held at the same potential. Since $\omega=0<\omega_d$, we know from the analysis in Sec. III that W is a local maximum for the cylindrically symmetric equilibrium and, correspondingly, that the eigenvalues λ_j in expansion (27) are all positive. Moreover, from Eq. (34) one can guess that for $\omega=0$ the two lowest eigenvalues $(\lambda_1=\lambda_2)$ are separated from zero by order unity. Since the asymmetric equilibrium differs only slightly from the symmetric equilibrium, one expects by continuity that the eigenvalues for the asymmetric equilibrium differ only slightly from those for the symmetric equilibrium. Thus the eigenvalues for the symmetric equilibrium are all positive, and W is a local maximum. We may conclude that the asymmetric equilibrium is stable to small-amplitude $\mathbf{E}\times\mathbf{B}$ drift perturbations. The fact that these configurations are local maxima of the electrostatic energy has a somewhat surprising consequence. The electron plasma is effectively *repelled* by a segment of the boundary, which is held at a positive potential or *attracted* by a negative one. This is consistent with behavior seen in experiments.[14]

Finally, all of the analysis in this paper is based on 2-D $\mathbf{E}\times\mathbf{B}$ drift dynamics, and some comments concerning the relevance of this approximation to real three-degrees-of-freedom plasmas are necessary. We have in mind that the magnetic field is sufficiently strong that the $\mathbf{E}\times\mathbf{B}$ drift rotation frequency (and the diocotron mode frequencies) are small compared to the characteristic bounce frequency for an electron. (This limit should not be hard to satisfy in experiments, although it is only marginally attained for the parameters of Ref. 4.) In this case, a theory of the low-frequency motion can be developed as an expansion in the inverse of the bounce frequency, and 2-D $\mathbf{E}\times\mathbf{B}$ drift dynamics arises in lowest order, that is, as bounce average dynamics. For the bounce average dynamics to reduce to true 2-D $\mathbf{E}\times\mathbf{B}$ dynamics, it is also necessary for the column to be sufficiently long that solutions to Poisson's equation are approximately 2-D.

In other words, the stability theorem should be interpreted as insurance against low-frequency (rotation frequency) instabilities. Also, when the axial bounce frequency is not large compared to these frequencies, the theorem cannot be trusted. For example, the bounce action is then not a good adiabatic invariant, and the parallel kinetic energy can vary, that is, can be interchanged with electrostatic energy. Of course, such a possibility vitiates the stability theorem.

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