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# Maximization of vortex entropy as an organizing principle in intermittent, decaying, two-dimensional turbulence

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The statistical mechanics of a discrete-vortex system is considered in a coarse-grained sense, as a model for relaxation of two-dimensional turbulence at very large Reynolds numbers. The implications are qualitatively consistent with observations from recent numerical simulations of decaying homogeneous turbulence.

Attempts to apply methods of statistical thermodynamics to turbulent flows have usually been based on wave number decomposition of an Eulerian velocity field.[1,2] Intermittency, which involves spatially localized phenomena and hence phase correlations, is notoriously difficult to treat in this way. An essential feature of turbulence is the randomization of positions of fluid elements, in the sense of increasing configuration entropy. In two-dimensional (2D) flow at very large Reynolds number, the total energy and the circulation of each fluid element are nearly conserved quantities. Accordingly, Onsager[3] based an alternative statistical approach on an idealization of Lagrangian fluid elements as point vortices. Montgomery[4] employed this approach to describe time-averaged properties of early numerical simulations of 2D turbulence, without reference to intermittency. In recent years, direct numerical simulation has been applied to decaying, incompressible, 2D turbulence obeying Navier-Stokes and related equations of motion at higher resolution and over longer time intervals;[5-8] the results furnish detailed information about the localized vortex structures which embody intermittent behavior. I will show that discrete-vortex statistics, when extended to incorporate time dependence, can account for several of the features observed in such simulations, namely the presence and predominant form of eddies, and some gross features of the structure function or energy spectrum. I extend earlier research (e.g., Refs. 9 and 10) by emphasizing fundamentally thermodynamic aspects of the evolution.

The vorticity evolves according to the Navier-Stokes equation

$$\frac{\partial \omega}{\partial t}+\mathbf{v}\cdot\nabla\omega=\nu\nabla^2\omega\,. \tag{1}$$

The velocity is given by $\mathbf{v}=-\hat{\mathbf{x}}\partial_y\psi+\hat{\mathbf{y}}\partial_x\psi$, where the stream function $\psi$ satisfies $\nabla^2\psi=-\omega$ with appropriate boundary conditions. My objective is an approximate description of decaying turbulence (i.e., small $\nu$) after initial transients but before asymptotic viscous decay. The energy $E=\frac{1}{2}\int d^2\mathbf{r}\,\omega\psi$ and the absolute circulation $\Gamma_{\rm abs}=\int d^2\mathbf{r}\,|\omega|$ will be regarded as constants. The precise form of dissipation terms acting on small scales is not crucial for intermediate and large-scale behavior, so many of the following arguments apply to generalizations of Eq. (1).[5]

The idea of maximizing entropy subject to a few conservation laws may be compared to a selective decay hypothesis,[11,12] which represents turbulent dynamics as minimizing a more rapidly decaying quantity (enstrophy, say) while nearly preserving others (energy). Such approaches lead to variational schemes which predict the typical forms of eddies and energy spectra. Applications of selective decay hypotheses suffer from the conceptual flaw that the choice of important dissipated functionals of vorticity is somewhat artificial.[13] One might regard the conservation of circulation of small fluid elements as an approximation to the conservation of *all* integrated functions of vorticity, which obtains in the nondissipative limit. While the circulations of fluid elements change only very slowly (provided there is little structure on the smallest spatial scales), dissipative effects or coarse-graining are more effective in breaking the severe topological constraints of ideal Euler flow. The time scale for the decay of the energy is prolonged by the tendency for energy to collect at large spatial scales in 2D turbulence.[14,15] The constrained randomization hypothesis leads us to study the statistics of the phase space for fluid elements.

Turbulent flow is accordingly modeled by a collection of discrete vortices, which idealize Lagrangian fluid elements.[16] Recent mathematical studies and numerical applications show that many-vortex dynamics approximates continuous fluid flow under a variety of conditions.[17] In such models, viscosity acts as a random force on the elements.[18] In the continuous fluid, viscosity imposes a minimum contributing scale size, such as $l_\nu=(\nu/\sup\omega)^{1/2}$, the diffusion length for the eddy turnover period at the maximum realized vorticity.[19] Thus a small viscosity avoids some delicate questions associated with the continuum limit of Euler flow. To the extent that viscosity is weak but nonvanishing, the vortex system may plausibly be modeled by a Hamiltonian system with many degrees of freedom but few constants of the motion, so that an ergodic hypothesis is a natural starting point. Let us further approximate the dynamics by replacing the vorticity distribution by a collection of point vortices. Following Onsager, I assume that the flow tends to ergodize the distribution of vorticity elements over the energy shell in their phase space, canonical coordinates for which are just the spatial coordinates of the elements.[20] This point-vortex system is the microscopic basis for a coarse-grained description, which should approximate the behavior of a continuous fluid with vorticity smoothed over a coarse-

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graining length $a \geq l_v$.

Let the coarse-grained densities of elementary vortices of circulations $\gamma$ and $-\gamma$ be $\rho_+$ and $\rho_-$, respectively. Consider the most probable densities, as described by Joyce and Montgomery.[21] The statistical weight of a given configuration is just the phase-space area, which is proportional to the exponential of an entropy approximated by $S\{\rho\} = -\int d^2\mathbf{r}\,\rho_+ \ln(\rho_+ a^2) + \rho_- \ln(\rho_- a^2)$. The problem of spatial degeneracy for an incompressible fluid has been circumvented by the point-particle idealization. For definiteness, I consider decaying turbulence in a periodic domain of size $L \times L$. The system is isolated, so a microcanonical ensemble is appropriate (at least if $a$ is comparable to $l_v$). Hence the mean-field equilibria $\{\rho_+, \rho_-\}$ are obtained by maximizing $S$ subject to the constraints of specified energy $E$ and signed circulations $N = \int d^2\mathbf{r}\,\rho_\pm$. For present purposes it is convenient to normalize time so that $\gamma = 1$.

The Euler-Lagrange equations satisfied by maximum-entropy configurations are $\ln \rho_\pm = \mu_\pm \mp \beta\psi$, where $\mu_\pm$ and $\beta$ are Lagrange multipliers ($\beta$ is formally an inverse temperature). Such equations ignore statistical fluctuations, so I will refer to them as "mean-field equations." Requiring that the total circulation vanish (in accordance with periodic boundary conditions) leads to the equilibrium vorticity $\bar{\omega} = -2e^{\mu}\sinh\beta\bar{\psi}$ [the sinh-Poisson equation (SPE)].[21] Level curves of vorticity and stream function coincide for solutions of the SPE, so they represent steady flows. There are multiple solutions for positive energies; these have been studied in Refs. 21–25. The structure and classification of the solution branches depends on the geometry. In cases with symmetry, there are typically bifurcations among stationary-entropy states at negative values of $\beta$, some of which may be interpreted as phase transitions.[26] In the periodic domain, the SPE has the trivial solution $\bar{\omega} = 0$ at any value of $\beta$, but vortical structure or nonzero mean-field energy requires that $\beta < 0$. The maximum-entropy solution consists of two eddies of opposite circulation. Other solutions (vortex-core lattices) are saddle points of the entropy, which have previously been dismissed as thermodynamically irrelevant. However, these may locally approximate the transient states arising as entropy increases from an arbitrary initial condition, since local rearrangement of fluid elements is more rapid than global relaxation. Global entropy maximization predicts that long-time averages (stationary statistical equilibrium) would be dominated overwhelmingly by the two-eddy configurations. They would be stationary in the field of their periodic images, but drift because of residual fluctuations. Of course, this prediction ignores the effects of viscosity on the time scales necessary to reach such a state; the eddies themselves will decay slowly to different forms.[27] The latter decay occurs on time scales beyond the reach of high-Reynolds-number simulations. It is more relevant for the present discussion that the entropy maximum may be inaccessible for dynamical reasons, as discussed below.

It is illustrative to consider the typical form of an intermediate-scale eddy using a Gibbsian approach and a simple dynamical argument. The simplest such coherent structure in this context is a distribution $\omega(\mathbf{r})$ of a subset of the vorticity elements, dominated by those of one sign and localized in a part of the domain. Boundary conditions corresponding to the (slowly varying) potential flow due to other, distant vortices are to be imposed far from the vortex core. First let us consider axisymmetric Dirichlet boundary conditions, which impose another constraint, conservation of angular impulse (essentially the canonical angular momentum) $M = \int d^2\mathbf{r}\, r^2\omega$, where $r$ is the distance from the center of the structure. In the limit that no elements of the other sign are present, the corresponding mean-field equation is $\omega = \exp[-\beta(\psi + \Omega r^2) + c]$, where the inverse temperature $\beta$, the rotation frequency $\Omega$ of an equilibrium frame, and the normalization constant $c$ arise as Lagrange multipliers.[9,22,23,26,28]

Now let the axisymmetric boundary conditions be perturbed by a weak multipole field which varies slowly compared with an eddy-turnover time; this mediates an exchange of angular momentum, which is the dominant interaction with distant regions. (Energy, which is dominated by the self-energies of eddies, relaxes much more slowly.) To be consistent with the prescribed vanishing total circulation these interactions drive each vortex towards a state with $\Omega = 0$. There is an exact solution[29] to the mean-field theory in this case—the squared-Lorentzian vortex[30] (SLV): $\omega(r) = \omega_0[1 + (r/l)^2]^{-2}$. These structures are described entirely in terms of an amplitude and a characteristic scale length $l$ determined by the temperature. The Fourier transform is $\tilde{\omega}(k) \propto kK_1(k)$, where $K_1$ is the modified Bessel function. The contribution to the energy spectrum from a single SLV is proportional to $k^{-1}$ at small $k$, but exponentially decreasing above a characteristic wave number. The SLV is consistent with the published plots of many structures seen in numerical experiments.[6,7]

The SLV approximates the core structures in the most probable (two-eddy) configurations of discrete vortices at sufficiently large energy. Hence truly ergodic evolution at large energies would lead to long-time averages with a *total* energy spectrum proportional to $k^{-1}$, up to a cutoff wave number $k \sim l^{-1}$ determined by the maximum vorticity and the total energy (i.e., independent of viscosity). This differs from the asymptotic $k^{-1}$ spectrum of absolute equilibrium for wave-number statistics,[15] in that there is no condensation in the lowest wave number; the nonlinear transfer of energy to large scales is inhibited by the tendency of vorticity contours to coincide with streamlines.

The SLV is only the simplest of a class of maximum-entropy structures, parameterized by the choice of bounding streamline and the relative concentrations of the two species. With other boundary conditions, entropy maximization leads to configurations such as Kelvin-Stuart cat's-eyes (a row of vortices in shear flow). Published data showing the functional dependence of vorticity on streamfunction in long-lived structures seen in experiments and simulations seem qualitatively consistent with the hyperbolic-sine or exponential profiles which follow from local entropy maximization in the neutral and single-sign cases, respectively. In the small-amplitude limit, circular maximum-entropy vorticity structures are approximately Bessel-function distributions.[12,31] Thus the maximization of entropy generalizes the minimization of enstrophy to cases exhibiting more sharply localized vorticity.

As indicated by Benzi *et al.*,[7] the later stages of evolution are probably best described in terms of the dynamics of distinct vortices. It is interesting to consider the scaling behavior which follows from a crude kinetic theory based on modeling these structures as SLV's between collisions.

Let us estimate the parameters which describe an SLV resulting from the collision of an earlier pair. In such a merger event, total energy, angular momentum, and generalized enstrophies are nearly conserved. During the interaction, filaments are produced which typically contain a small fraction of the total circulation; these can move to arbitrarily large distances from the center of circulation, so as to conserve angular momentum without constraining the form of the remaining core. In the spirit of our coarse-grained description, the generalized enstrophies may also be regarded as unconstrained. The dominant contributions to the energy, on the other hand, are from the core self-energies before and after the collision. If we limit our attention to structures significantly larger than the coarse-graining scale, the maximum vorticity is also nearly unchanged. For simplicity, let us suppose that all vortices have the same peak vorticity. (This is consistent with initial conditions leading to nucleation around nearly identical, well localized, initial vorticity extrema.[7])

The SLV vorticity can be written as

$$\omega(\Gamma,l;r)=\frac{\Gamma l^2}{\pi[l^2+(1-l^2)r^2]^2}\,, \qquad (2)$$

where $\Gamma$ is its circulation, and $l$ is a characteristic radius measured in units of a system scale $L$. Its self-energy is

$$E(\Gamma,l)=\frac{\Gamma^2}{8\pi(1-l^2)^2}(-2\ln l-1+l^2)\,. \qquad (3)$$

($E$ and $\Gamma$ are evaluated as integrals over a disk concentric with the core, with radius $L$ comparable to the system size; weak dependencies on this radius will be ignored.) The SLV is assumed to be well localized, so $l\ll 1$. Now suppose two SLV's with parameters $l_j,\Gamma_j$, $j=1,2$ collide to form one with $l_f,\Gamma_f$ and filaments far from the core. The central vorticities are equal by assumption, so $\Gamma_j/l_j^2$ is independent of $j$. Conservation of core energy is expressed by $\hat{E}(l_1)+\hat{E}(l_2)=\hat{E}(l_f)$, where $\hat{E}(x)=-x^4(1+2\ln x)$. If pairwise collisions are the dominant process (this is more general and more realistic than the hierarchy postulated in Ref. 7), then the distribution function $n(l,t)$ obeys

$$\frac{\partial n(l,t)}{\partial t}=\int dl'dl''W(l',l'';l,t)-W(l,l'';l';t)\,. \qquad (4)$$

The first term represents formation of eddies of size $l$ from pairs of smaller ones, the second coalescence with eddies of any size. The transition rate $W(x,y;z;t)$ is assumed to be concentrated on the surface $\hat{E}(x)+\hat{E}(y)=\hat{E}(z)$, which defines an implicit function $Y(x,z)$. $W$ then presumably has the form

$$W(x,y;z;t)=n(x,t)n(y,t)\overline{\sigma v}\delta(y-Y(x,z))\,, \qquad (5)$$

where $\overline{\sigma v}$ is an average product of velocity and cross section, which depends on $x$ and $y$. We can use the characteristic radius and the (nearly constant) rms velocity $\bar{v}$ to approximate $\overline{\sigma v}$ by $\bar{v}L[\max(x,y)]^{1/2}$. Then

$$\frac{\partial n(l,t)}{\partial t}=L\bar{v}\int_0^l dx\,n(x,t)n[Y(x,l),t][\max(x,Y(l,x))]^{1/2} -L\bar{v}n(l,t)\int_0^1 dx\,n(x,t)[\max(x,l)]^{1/2}\,. \qquad (6)$$

Dimensional analysis of Eq. (6) suggests that $n(l,t)\propto\hat{n}(l)t^{-1}$, and if production of vortices of size $l$ from smaller ones predominates over absorption into larger ones, then a scaling ansatz $[\hat{n}(l)\propto l^b]$ yields $b=-2$, which is consistent with the observations in Ref. 7. The energy spectrum of a single vortex is approximately $A(l)k^{-1}H(l-k^{-1})$ where $H$ is the heaviside function. For structures with the same peak vorticity $\omega_0$, this implies that $A(l)\approx 2\omega_0^2 l^4$, so that the total contribution to the energy spectrum from localized vortices is

$$E(t,k)\approx\int_0^{1/k}dl\,A(l)k^{-1}n(t,l)\propto t^{-1}k^{-4}\,. \qquad (7)$$

This spectrum[32] is augmented by contributions from small-amplitude, wavelike fluctuations spread over large regions. Merger events will produce filamentary structures which may undergo Kelvin-Helmholtz instabilities and generate more small vortices (with peak vorticity differing from $\omega_0$). Eventually, most of the vorticity will be contained in a few large vortices. Their arrangement may be close to a stable inviscid equilibrium, which would invalidate the ergodic hypothesis. In view of the crudeness of this theory and the simplifying assumptions, more detailed comparison between the underlying ideas and numerical experiments is desirable.

The relaxation process depends on the initial conditions and on the character of any externally imposed strain fields or vorticity sources. I assumed above that the dynamics can be represented (at least statistically) by the ergodic interaction of point vortices. This is most plausible for initial conditions with a structure function $\langle\omega^2\rangle_k$ peaked at very large wave numbers. When the circulation is more broadly distributed, a better description might be obtained by generalizing the entropy functional $S$ to treat the statistics of nonoverlapping fluid elements with arbitrary vorticity.[33]

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