Transport in a toroidally confined pure electron plasma

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O'Neil and Smith [T.M. O'Neil and R.A. Smith, Phys. Plasmas **1**, 8 (1994)] have argued that a pure electron plasma can be confined stably in a toroidal magnetic field configuration. This paper shows that the toroidal curvature of the magnetic field of necessity causes slow cross-field transport. The transport mechanism is similar to magnetic pumping and may be understood by considering a single flux tube of plasma. As the flux tube of plasma undergoes poloidal **E×B** drift rotation about the center of the plasma, the length of the flux tube and the magnetic field strength within the flux tube oscillate, and this produces corresponding oscillations in $T_\parallel$ and $T_\perp$. The collisional relaxation of $T_\parallel$ toward $T_\perp$ produces a slow dissipation of electrostatic energy into heat and a consequent expansion (cross-field transport) of the plasma. In the limit where the cross section of the plasma is nearly circular the radial particle flux is given by $\Gamma_r = \frac{1}{2}\nu_{\perp,\parallel} T(r/\rho_0)^2 n/(-e\partial\Phi/\partial r)$, where $\nu_{\perp,\parallel}$ is the collisional equipartition rate, ρ_0 is the major radius at the center of the plasma, and r is the minor radius measured from the center of the plasma. The transport flux is first calculated using this simple physical picture and then is calculated by solving the drift-kinetic Boltzmann equation. This latter calculation is not limited to a plasma with a circular cross section. © *1996 American Institute of Physics.* [S1070-664X(96)01407-3]

I. INTRODUCTION

Pure electron plasmas confined by a toroidal magnetic field have been studied both experimentally and theoretically since the 1960s[1–3] and have received renewed attention in recent years.[4–6] The equilibria of pure electron plasmas confined by a toroidal magnetic field were studied by Dougherty and Levy.[2] The equilibria exist due to the strong space charge electric fields that arise because the plasma is nonneutral. These fields cause particle drift orbits to be closed. One can think of the poloidal **E×B** drifts as providing the rotational transform.

Recently, O'Neil and Smith[5] argued that a pure electron plasma can be confined stably in such a configuration when the frequencies are ordered so that the cross-field motion may be described by toroidal-averaged drift dynamics. They found equilibria for which the energy is a maximum relative to neighboring states. The system point evolves on a contour of constant energy in the space of accessible states, and when the energy is a maximum, the contour shrinks to a point and no further change in the state is possible.

In this paper we obtain a collisional transport equation for a pure electron plasma that is confined in this geometry. We assume the same frequency ordering that was used to analyze stability[5]

$$\Omega_c \gg \bar{\omega}_T \gg \omega_E \gg \nu \gg \tau^{-1}, \tag{1}$$

where Ω_c is the cyclotron frequency, $\bar{\omega}_T = \bar{v}/\rho$ is the toroidal rotation frequency for a thermal particle, $\omega_E \sim \omega_p{}^2/\Omega_c$ is the characteristic **E×B** drift frequency in the poloidal direction, ν is the collision frequency, and τ is the transport time scale. The length scale ordering is

$$\rho_0 \gg r \gg \lambda_D \tag{2}$$

where ρ_0 is the major radius at the center of the plasma, r is the minor radius measured from the center of the plasma, and λ_D is the Debye length. We also assume that $r^2\omega_E{}^2/c^2 \ll 1$ so that the diamagnetic corrections to the magnetic field are negligible. These conditions are well satisfied in typical experiments.

There are two ways to understand the transport. One can focus on a flux tube and note that the length of the tube and the magnetic field strength in the tube vary as the tube undergoes poloidal **E×B** drift rotation. The constancy of the adiabatic invariants $\mu = mv_\perp{}^2/2B$ and $I = (2\pi)^{-1}\oint dl\, mv_\parallel$ then imply a cyclic variation in $T_\parallel$ and $T_\perp$. The variations are unequal, and collisional relaxation between $T_\parallel$ and $T_\perp$ produces a slow heating of the plasma. The process is similar to magnetic pumping[7] and to rotational pumping.[8] This heating comes about at the expense of electrostatic energy, so the plasma must expand in minor radius. In section II, we use this viewpoint to calculate the radial flux for the simple case where the plasma has circular cross section.

Alternatively, one can focus directly on the drift orbits as determined by the particle energy H and the adiabatic invariants μ and I. When a particle undergoes velocity scattering in a collision, these quantities change value and the drift orbit changes, allowing the particle to step in radius. In section III, the drift kinetic Boltzmann equation is used to calculate the flux. This calculation does not require the plasma cross section to be circular but reduces to the result of section II when a circular cross section is specified.

II. HEATING AND TRANSPORT

A schematic diagram for a toroidal trap is shown in Fig. 1. The confinement region is bounded by a toroidal conductor and the magnetic field is purely toroidal. Here (ρ,Θ,z) is

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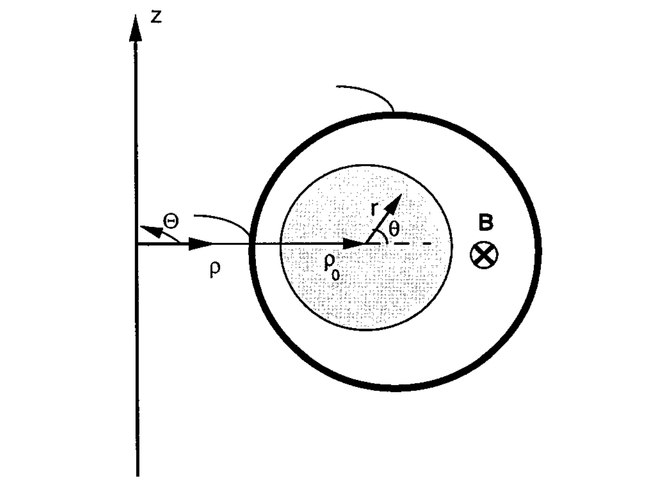

FIG. 1. Toroidal confinement geometry, with the coordinates (ρ,Θ,z) and (r,θ) illustrated.

a cylindrical coordinate where ρ is the major radius, Θ is the toroidal angle, and the z axis is the axis of symmetry of the torus.

In this section we assume for simplicity that the plasma has a circular cross section. This assumption is not necessary and will be relaxed in section III. For the case of a circular cross section it is useful to introduce a polar coordinate system (r,θ) which is centered on the plasma and is locally oriented perpendicular to the magnetic field. Here θ is the poloidal angle and r is the minor radius measured from the center of the plasma, The (ρ,Θ,z) coordinate system and the (r,θ) coordinate system are related through the relations

$$\rho=\rho_0+r\cos\theta, \tag{3}$$

$$z=r\sin\theta, \tag{4}$$

where ρ_0 is the major radius at the center of the plasma.

We derive an expression for the radial particle flux by considering a single flux tube of plasma as shown in Fig. 2. The flux tube has length $L(r,\theta)=2\pi\rho(r,\theta)$, cross-sectional area δA, and contains δN particles, where δN is a constant. Using Ampère's law, the toroidal magnetic field can be expressed as $\mathbf{B}=\hat{\Theta}B_0\rho_0/\rho$ where B_0 and ρ_0 are constants. Thus, the field strength in the flux tube is $B_0\rho_0/\rho(r,\theta)$, where $\rho(r,\theta)$ is given by Eq. (3). The dominant motion of the flux tube is the $\mathbf{E}\times\mathbf{B}$ drift. Under the assumption of a small inverse aspect ratio $(r/\rho_0 \ll 1)$ the electric field is nearly radial and the flux tube drifts in a circular orbit with frequency

$$\omega_E=\frac{c}{Br}\frac{\partial\Phi(r)}{\partial r}. \tag{5}$$

As the flux tube drifts toward the inside of the torus its length decreases and the magnetic field inside the flux tube increases. Setting $\theta=\omega_E t$ and using $r/\rho_0 \ll 1$ yields

$$\begin{aligned} L(t)&=2\pi\rho_0+2\pi r\cos\omega_E t,\\ B(t)&=B_0-B_0\frac{r}{\rho_0}\cos\omega_E t. \end{aligned} \tag{6}$$

Since the magnetic moment, $\mu=mv_\perp^2/2B$, is a constant, the perpendicular velocity of each particle increases as the magnetic field increases. Of course, the average magnetic moment is also constant and is related to the perpendicular temperature by

$$\text{const.}=\frac{1}{\delta N}\sum_{i=1}^{\delta N}\frac{\frac{1}{2}mv_{\perp i}^2}{B}=\frac{1}{B}T_\perp. \tag{7}$$

Differentiating this equation with respect to time yields an equation for the perpendicular temperature evolution

$$\frac{\partial T_\perp}{\partial t}=\frac{1}{B}\frac{\partial B}{\partial t}T_\perp. \tag{8}$$

Similarly, the average of the square of the individual toroidal actions is a constant and is related to the parallel temperature by

$$\text{const.}=\frac{1}{\delta N}\sum_{i=1}^{\delta N}(Lmv_i)^2=\frac{2}{m}L^2T_\parallel. \tag{9}$$

Differentiating this expression with respect to time yields

$$\frac{\partial T_\parallel}{\partial t}=-\frac{2}{L}\frac{\partial L}{\partial t}T_\parallel. \tag{10}$$

The parallel and perpendicular temperatures also couple collisionally so that the full temperature evolution is more accurately described by

$$\frac{\partial T_\parallel}{\partial t}=-\frac{2}{L}\frac{\partial L}{\partial t}T_\parallel+2\nu_{\perp,\parallel}(T_\perp-T_\parallel) \tag{11}$$

and

$$\frac{\partial T_\perp}{\partial t}=\frac{1}{B}\frac{\partial B}{\partial t}T_\perp-\nu_{\perp,\parallel}(T_\perp-T_\parallel), \tag{12}$$

where $\nu_{\perp,\parallel}$ is the equipartition rate. The factor of 2 difference in the collisional coupling term for Eq. (11) relative to Eq. (12) simply reflects the fact that there are two perpendicular degrees of freedom and one parallel.

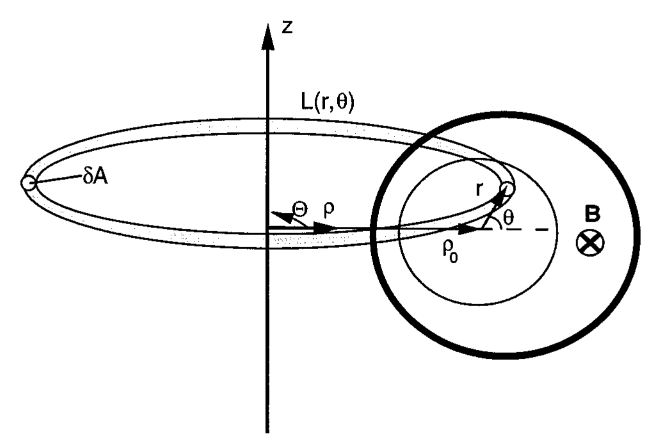

FIG. 2. A flux tube of plasma.

A two time scale analysis of these equations based on the smallness of r/ρ_0 and on the frequency ordering $\omega_E \gg \nu$ yields the result

$$\frac{d[\frac{1}{2}\langle T_\parallel \rangle + \langle T_\perp \rangle]}{dt} = \frac{1}{2}\nu_{\perp,\parallel}\left(\frac{r}{\rho_0}\right)^2 T, \tag{13}$$

where $\langle \cdot \rangle$ indicates an average over the fast time scale, that is, over a poloidal **E×B** drift time. The heating of the plasma arises because the parallel and perpendicular temperature fluctuations are unequal. Collisions cause a small phase shift in the fluctuations and to second order in r/ρ_0 there is a net heating in the plasma.

Since the confinement potentials are time independent, the total energy in the plasma is conserved and the increase in thermal energy must be balanced by a corresponding decrease in the electrostatic energy. The particle flux is found by equating the increase in thermal energy to local Joule heating

$$n\frac{d}{dt}\left(\frac{1}{2}\langle T_\parallel \rangle + \langle T_\perp \rangle\right) = -e\frac{\partial \Phi}{\partial r}\Gamma_r, \tag{14}$$

where Γ_r is the radial particle flux and n is the density. The right hand side of this equation is the Joule heating per unit volume. Equations (13) and (14) are solved for the flux and yield

$$\Gamma_r = \frac{1}{2}\nu_{\perp,\parallel} n(r)\frac{T}{-e\,\partial\Phi/\partial r}\left(\frac{r}{\rho_0}\right)^2. \tag{15}$$

Note that the flux depends on magnetic field strength only through $\nu_{\perp,\parallel}$. This rather surprising result is due to an accidental cancellation. The net heating in each poloidal rotation is proportional to the phase shift in the temperature fluctuations which scales as $\nu_{\perp,\parallel}/\omega_E$. The heating rate is equal to the heating per poloidal rotation times the poloidal rotation frequency. Therefore, ω_E drops out of the calculation. In the regime of weak magnetization (i.e., $r_c \gg b$, where $r_c = \bar{v}/\Omega_c$ and $b = e^2/m\bar{v}^2$), the dependence on the magnetic field strength is very weak, $\nu_{\perp,\parallel} \propto \ln(r_c/b)$. In the regime of strong magnetization (i.e., $r_c \ll b$) $\nu_{\perp,\parallel}$ becomes exponentially small[8–10] and our theory predicts that Γ_r becomes exponentially small.

III. KINETIC TREATMENT

One can also understand this transport process in terms of single particle drift orbits. The frequency ordering $\Omega_c \gg \bar{\omega}_T \gg \omega_E$ ensures that the magnetic moment (cyclotron action) $\mu = mv_\perp^2/2B$ and the toroidal action

$$P_\Theta = \frac{1}{2\pi}\oint P_\Theta d_\Theta = \frac{1}{2\pi}\oint mv_\parallel dl \tag{16}$$

are good adiabatic invariants. Since the confinement potentials are time independent, the energy is also conserved. The closed single particle drift orbits are determined by these three constants. As illustrated in Fig. 3, when a particle undergoes a collision its parallel and perpendicular velocity change, and it begins to move on a new drift surface. This event constitutes the fundamental step underlying the transport.

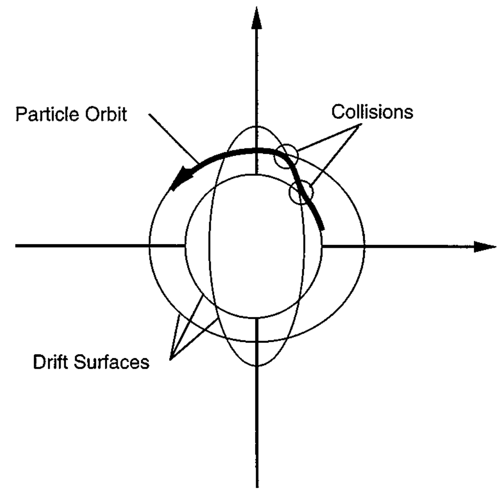

FIG. 3. An illustration of how collisions allow a particle to change drift surfaces.

The guiding center Hamiltonian in toroidal geometry is given by[11]

$$H = \frac{P_\Theta^2}{2m\rho^2(p_z)} + \frac{\mu B_0 \rho_0}{\rho(p_z)} + e\Phi(z,\rho(p_z)), \tag{17}$$

where z and p_z are canonically conjugate, $p_z = (eB_0\rho_0/c)\ln(\rho_0/\rho)$, and ρ_0 and B_0 are constants. The first term gives the curvature drift, the second the gradient $|B|$ drift and the third term the **E×B** drift. In a pure electron plasma the **E×B** drift is the dominant drift. Equivalently, one may say that the poloidal drift surfaces differ only slightly from the equipotential surfaces.

It is useful to introduce a canonical transformation

$$\begin{aligned} z &= z(p_\psi, \psi), \\ p_z &= p_z(p_\psi, \psi), \end{aligned} \tag{18}$$

which is chosen such that

$$\Phi(z,\rho(p_z)) = \Phi(p_\psi). \tag{19}$$

The new momentum, p_ψ, is nearly constant during the evolution except for small curvature and gradient $|B|$ drifts. The Hamiltonian takes the form

$$H = \frac{P_\Theta^2}{2m\rho^2(p_\psi,\psi)} + \frac{\mu B_0 \rho_0}{\rho(p_\psi,\psi)} + e\Phi(p_\psi). \tag{20}$$

The gradient $|B|$ and curvature drifts normal to the $p_\psi =$ const. surfaces are proportional to $\partial H/\partial \psi$.

We represent the plasma with a distribution of guiding centers

$$f = f(p_\Theta, \Theta, p_\psi, \psi, \mu, t). \tag{21}$$

This distribution evolves according to the drift-kinetic Boltzmann equation

$$\frac{\partial f}{\partial t} + [f,H] = C(f), \tag{22}$$

where $C(\cdot)$ is the collision operator and the Poisson bracket is given by

$$[f,H]=\frac{\partial f}{\partial\Theta}\frac{\partial H}{\partial p_\Theta}+\frac{\partial f}{\partial\psi}\frac{\partial H}{\partial p_\psi}-\frac{\partial f}{\partial p_\psi}\frac{\partial H}{\partial\psi}. \tag{23}$$

For our frequency ordering $\omega_T=\partial H/\partial p_\Theta$ is large and so $\partial f/\partial\Theta$ must be small. Physically this corresponds to the fact that any initially large Θ variations are rapidly mixed by the toroidal streaming along the magnetic field. The small Θ variations are uninteresting from the standpoint of cross-field transport and may be eliminated by integrating Eq. (22) over Θ, that is averaging over the toroidal motion. The result is

$$\frac{\partial\bar{f}}{\partial t}+\frac{\partial\bar{f}}{\partial\psi}\frac{\partial H}{\partial p_\psi}-\frac{\partial\bar{f}}{\partial p_\psi}\frac{\partial H}{\partial\psi}=C(\bar{f}), \tag{24}$$

where

$$\bar{f}(p_\Theta,p_\psi,\psi,\mu,t)=\int_0^{2\pi}d\Theta\, f(p_\Theta,\Theta,p_\psi,\psi,\mu,t). \tag{25}$$

Rewriting this equation as

$$\frac{\partial\bar{f}}{\partial t}+\frac{\partial}{\partial\psi}\left[\frac{\partial H}{\partial p_\psi}\bar{f}\right]-\frac{\partial}{\partial p_\psi}\left[\frac{\partial H}{\partial\psi}\bar{f}\right]=C(\bar{f}) \tag{26}$$

and integrating over p_Θ, μ, and ψ yields a transport equation

$$\frac{\partial N(p_\psi)}{\partial t}=\frac{\partial}{\partial p_\psi}\left[\int\frac{d\psi}{2\pi}\int dp_\Theta d\mu\frac{\partial H}{\partial\psi}\bar{f}\right], \tag{27}$$

where

$$N(p_\psi)=\int\frac{d\psi}{2\pi}\int dp_\Theta d\mu\bar{f}. \tag{28}$$

The integral over the collision operator vanishes because collisions conserve the number of particles.

To obtain a transport equation accurate to second order in the small quantity, $\partial H/\partial\psi$, we need only obtain $\bar{f}$ accurate to first order in $\partial H/\partial\psi$. Thus we look for a solution to Eq. (24) in the form

$$\bar{f}=\bar{f}_0(H,p_\psi)+\delta\bar{f}(p_\Theta,p_\psi,\psi,\mu), \tag{29}$$

where $\delta\bar{f}/\bar{f}_0\sim(1/H)(\partial H/\partial\psi)$ and

$$\bar{f}_0=N(p_\psi)(2\pi T/m)^{-3/2}e^{-H/T}e^{e\Phi/T}. \tag{30}$$

Written in velocity variables, $\bar{f}_0$ is just a Maxwellian times a density distribution which is constant on the equipotential contours. The linearized form of Eq. (24) is

$$\omega_E\frac{\partial\delta\bar{f}}{\partial\psi}+\frac{1}{\rho}\frac{\partial\rho}{\partial\psi}\left[2\frac{P_\Theta^2}{2m\rho^2}+\frac{\mu B_0R}{\rho}\right]\frac{\omega_E}{T}\bar{f}_0=C(\bar{f}_0+\delta\bar{f}), \tag{31}$$

where $\omega_E=\partial e\Phi/\partial p_\psi$ is the poloidal $\mathbf{E}\times\mathbf{B}$ drift frequency, and we have neglected terms of order $(1/N)\partial N/\partial p_\psi$ relative to ω_E/T because they are smaller by a factor $\lambda_D{}^2/r^2$.

Given the frequency ordering $\nu\ll\omega_E$, this equation may be solved perturbatively in the effective collision frequency. Dropping the collision operator and integrating yields

$$\delta\bar{f}^{(0)}=-\frac{\delta\rho}{\rho}\left[2\frac{P_\Theta^2}{2m\rho^2}+\frac{\mu B_0R}{\rho}\right]\frac{1}{T}\bar{f}_0 \tag{32}$$

where the superscript indicates the ordering in collisions and $\delta\rho$ is the ψ-dependent part of $\rho(p_\psi,\psi)$. To find the collisional response we insert $\delta\bar{f}^{(0)}$ into the collision operator on the right hand side of Eq. (31) and obtain

$$\frac{\partial}{\partial\psi}\delta\bar{f}^{(1)}=\frac{1}{\omega_E}C\left[\bar{f}_0\left(1-\frac{\delta\rho}{\rho}\left[2\frac{P_\Theta^2}{2m\rho^2}+\frac{\mu B_0R}{\rho}\right]\frac{1}{T}\right)\right]. \tag{33}$$

Substituting $\bar{f}_0+\delta\bar{f}^{(0)}+\delta\bar{f}^{(1)}$ into the transport equation yields

$$\frac{\partial N(p_\psi)}{\partial t}=\frac{\partial}{\partial p_\psi}\left[\int\frac{d\psi}{2\pi}\int dp_\Theta d\mu\left[2\frac{P_\Theta^2}{2m\rho^2}+\frac{\mu B_0R}{\rho}\right]\times\left(-\frac{1}{\rho}\frac{\partial\rho}{\partial\psi}\right)\delta\bar{f}^{(1)}\right]. \tag{34}$$

The collisionless terms vanish in the integral over ψ. Integrating by parts and substituting from Eq. (33) yields the result

$$\frac{\partial N(p_\psi)}{\partial t}=\frac{\partial}{\partial p_\psi}\left\{\int\frac{d\psi}{2\pi}\int dp_\Theta d\mu\left[2\frac{P_\Theta^2}{2m\rho^2}+\frac{\mu B_0R}{\rho}\right]\left(\frac{\delta\rho}{\rho}\right)\times\frac{1}{\omega_E}C\left[\bar{f}_0\left(1-\frac{\delta\rho}{\rho}\left[2\frac{P_\Theta^2}{2m\rho^2}+\frac{\mu B_0R}{\rho}\right]\frac{1}{T}\right)\right]\right\}. \tag{35}$$

After changing variables of integration from (p_Θ,μ) to $(v_\parallel,v_\perp)$ this may be written as

$$\frac{\partial N(p_\psi)}{\partial t}=\frac{\partial}{\partial p_\psi}\left\{\int\frac{d\psi}{2\pi}\int dv_\parallel d^2v_\perp\left[mv_\parallel{}^2+\frac{1}{2}mv_\perp{}^2\right]\left(\frac{\delta\rho}{\rho}\right)\times\frac{1}{\omega_E}C\left[\bar{f}_0\left(1-\frac{\delta\rho}{\rho}\left[mv_\parallel{}^2+\frac{1}{2}mv_\perp{}^2\right]\frac{1}{T}\right)\right]\right\}. \tag{36}$$

where $\bar{f}_0=N(p_\psi)f_M$ and f_M is a Maxwellian.

We take the collision operator in the general form

$$C(f)=\int d^3v_1d\sigma|v_{\rm rel}|(f(v_1')f(v')-f(v_1)f(v)), \tag{37}$$

where $d\sigma$ is the differential cross section and $v_{\rm rel}=v-v_1$. Using this form we obtain

$$\begin{aligned}\frac{\partial N(p_\psi)}{\partial t}=&-\frac{\partial}{\partial p_\psi}\left\{\int\frac{d\psi}{2\pi}\left(\frac{\delta\rho}{\rho}\right)^2\frac{N}{\omega_ET}N\right.\\&\times\int d^3v\left(mv_\parallel{}^2+\frac{1}{2}mv_\perp{}^2\right)\int d^3v_1d\sigma|v_{\rm rel}|\\&\times\left[\left(mv_\parallel{}'^2_1+\frac{1}{2}mv_\perp{}'^2_1+mv_\parallel{}'^2+\frac{1}{2}mv_\perp{}'^2\right)\right.\\&\times f'_{M1}f'_M-\left(mv_{\parallel1}^2+\frac{1}{2}mv_{\perp1}^2+mv_\parallel^2\right.\\&\left.\left.\left.+\frac{1}{2}mv_\perp{}^2\right)f_{M1}f_M\right]\right\}.\end{aligned} \tag{38}$$

Energy is conserved in the binary collisions. That is,

$$\frac{1}{2}mv_1^2+\frac{1}{2}mv^2=\frac{1}{2}mv'^2_1+\frac{1}{2}mv'^2. \tag{39}$$

This fact may be used to simplify Eq. (38) to the form

$$\frac{\partial N(p_\psi)}{\partial t}=-\frac{\partial}{\partial p_\psi}\left\{\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle_\psi\frac{N}{\omega_E}TN\int d^3v\left(\frac{1}{2}mv_\parallel{}^2\right)\right.$$
$$\times\int d^3v_1 d\sigma|v_{\rm rel}|\left[\left(\frac{1}{2}mv_\parallel{}'^2_1+\frac{1}{2}mv_\parallel{}'^2\right)f'_{M1}f'_M\right.$$
$$\left.\left.-\left(\frac{1}{2}mv_{\parallel 1}^2+\frac{1}{2}mv_\parallel{}^2\right)f_{M1}f_M\right]\right\}. \tag{40}$$

To evaluate the velocity integral in this equation, it is instructive to consider the collisional relaxation of an anisotropic Maxwellian distribution

$$f_A(v)=\left(\frac{2\pi T_\parallel}{m}\right)^{-1/2}\left(\frac{2\pi T_\perp}{m}\right)^{-1}\exp\left[-\frac{\frac{1}{2}mv_\parallel{}^2}{T_\parallel}-\frac{\frac{1}{2}mv_\perp^2}{T_\perp}\right]. \tag{41}$$

The change in the parallel temperature due to collisions is given by

$$\frac{d}{dt}\left(\frac{T_\parallel}{2}\right)=N\int d^3v\left(\frac{1}{2}mv_\parallel{}^2\right)\int d^3v_1 d\sigma|v_{\rm rel}|$$
$$\times(f_A(v_1')f_A(v')-f_A(v)f_A(v_1))$$
$$=\nu_{\perp,\parallel}(T_\perp-T_\parallel), \tag{42}$$

where the last line is used as a definition of the equipartition rate, $\nu_{\perp,\parallel}$. By substituting $T_\perp=T$ and $T_\parallel=(1-\alpha)T$ into the first line and taking the limit $\alpha\rightarrow 0$, one can easily show that

$$N\int d^3v\left(\frac{1}{2}mv_\parallel{}^2\right)\int d^3v_1 d\sigma|v_{\rm rel}|\left[\left(\frac{1}{2}mv_\parallel{}'^2_1+\frac{1}{2}m_\parallel{}'^2\right)\right.$$
$$\left.\times f'_{M1}f'_M-\left(\frac{1}{2}mv_{\parallel 1}^2+\frac{1}{2}mv_\parallel{}^2\right)f_{M1}f_M\right]$$
$$=-T^2\nu_{\perp,\parallel}. \tag{43}$$

This is precisely the integral that appears in Eq. (40). The transport equation can now be written in the relatively simple form

$$\frac{\partial N(p_\psi)}{\partial t}=-\frac{\partial}{\partial p_\psi}\left\{\nu_{\perp,\parallel}\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle_\psi\left(\frac{T}{-\omega_E}\right)N(p_\psi)\right\}. \tag{44}$$

This equation describes transport in poloidal action-angle variables. To make contact with section II, we consider the simple case where the plasma cross section is circular. In this case, $\psi=\theta$, $p_\psi=p_\theta=(eB/2c)r^2$, and

$$\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle_\psi=\int\frac{d\theta}{2\pi}\frac{r^2}{\rho^2}\cos^2\theta=\frac{r^2}{2\rho^2}. \tag{45}$$

The transport equation then becomes

$$\frac{\partial N(r)}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial r}r\left\{\frac{1}{2}\nu_{\perp,\parallel}\frac{T}{-e\partial\Phi/\partial r}\left(\frac{r}{\rho_0}\right)^2\right\}. \tag{46}$$

The quantity in brackets is the same radial particle flux that was found in section II.

ACKNOWLEDGMENTS

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